Rational Turnover Aversion: How Much Should a Portfolio Be Shrunk?*

Chulwoo Han†

Abstract

I develop a portfolio model that maximizes utility incorporating aversion to turnover from a reference portfolio. Irrational turnover aversion becomes rational behavior in the uncertain world. The usual expected utility loss minimization is sub-optimal due to model parameter uncertainty and is improved substantially when augmented with turnover aversion. Minimizing utility loss under high uncertainty requires an extreme degree of turnover aversion. The equal-weight portfolio serves better as the reference portfolio than the current portfolio even in the presence of transaction costs. The former renders less volatile portfolios and incurs lower transaction costs. This contradicts the models accounting for transaction costs in optimization. Proposed models significantly outperform various existing portfolio models.

JEL Classification: G11

Keywords: Optimal portfolio; Turnover aversion; Shrinkage estimator; Parameter uncertainty; Estimation error; Transaction cost

1 Introduction

The sensitivity of optimal portfolio models to the input parameters has long been plaguing both academics and practitioners, and is one of the crucial reasons behind the slow adoption of the models by the industry. Combined with the inevitable uncertainties in the input

*This paper is accompanied by an internet appendix (Internet Appendix). The previous version of this paper was entitled "Too Much Is Better than Optimal: Optimal Portfolio Choice under Parameter Uncertainty and Transaction Cost."

†Chulwoo Han is with Durham University. E-mail: chulwoo.han@durham.ac.uk.

parameters, an optimal portfolio could turn out to be disastrous and investors' reluctance to adopt optimal portfolio models may as well be considered rational behavior.

There has been a considerable amount of effort dedicated to address estimation errors and model sensitivity. One axis has been formed by the Bayesian approach: *e.g.*, Klein and Bawa (1976); Brown (1976, 1978); Jorion (1986); Black and Litterman (1992); Pástor (2000); Pástor and Stambaugh (2000) among others. For a review of Bayesian models, the reader is referred to Avramov and Zhou (2010). More recently, robust optimization that optimizes the portfolio under a worst-case scenario became popular: *e.g.*, Goldfarb and Iyengar (2003); Fabozzi et al. (2007); Ceria and Stubbs (2016). Kan and Zhou (2007) and Tu and Zhou (2011) optimally combine two or more portfolios so that the expected utility loss is minimized.[1] Incorporating transaction costs is also known to help reduce the sensitivity and improve the performance after transaction costs: *e.g.*, Gârleanu and Pedersen (2013); DeMiguel et al. (2015); Olivares-Nadal and DeMiguel (2015). Other approaches include imposing weight constraints (Jagannathan and Ma, 2003) or using a shrinkage method for parameter estimation (Ledoit and Wolf, 2004).

While these models claim to alleviate the problems arising from parameter uncertainty and perform better than the classical mean-variance model, DeMiguel et al. (2009) show that none of the portfolio models considered in their paper consistently outperforms the naïve, equal-weight portfolio. Their work triggered many researches that challenge the equal-weight portfolio and claim to outperform it: *e.g.*, Tu and Zhou (2011); Kirby and Ostdiek (2012); Bessler et al. (2014); Han (2016). Their evaluation method which compares risky portfolios derived from optimal portfolios has also been criticized for being unfair to some models (Kan et al., 2016). Still, most optimal portfolios seem to struggle to outperform the equal-weight portfolio consistently across assets and time.

Portfolio models that account for parameter uncertainty face two arduous challenges. One challenge arises from the quantification of uncertainty. The exact distributions of the parameter estimates are unknown and difficult to estimate. This problem is more severe if the estimates involve subjective elements such as analyst forecasts. Another challenge is that model parameters are usually nonlinear functions of unknown input parameters and also need to be estimated. Model parameter estimation is a nontrivial problem and a naïve plug-in method using input parameter estimates is likely to yield biased estimates. These issues make the behavior of the optimal portfolio somewhat unpredictable and often result in poor performance. Although the utility loss due to model parameter uncertainty can be potentially large, existing models (*e.g.*, Tu and Zhou, 2011; DeMiguel et al., 2015; Kan et al., 2016) have ignored it.

[1] This type of models are referred to as a shrinkage estimator.

If the input and model parameters are subject to uncertainty, the investor would be reluctant to invest in the optimal portfolio and willing to sacrifice a fraction of (*ex-ante*) utility to have a more robust portfolio. Even without estimation errors, investors are psychologically reluctant to extreme turnover, let alone the high transaction costs it involves. To reflect this type of behavior in a portfolio choice problem, I introduce *turnover aversion utility* that penalizes turnover from a reference portfolio.[2]

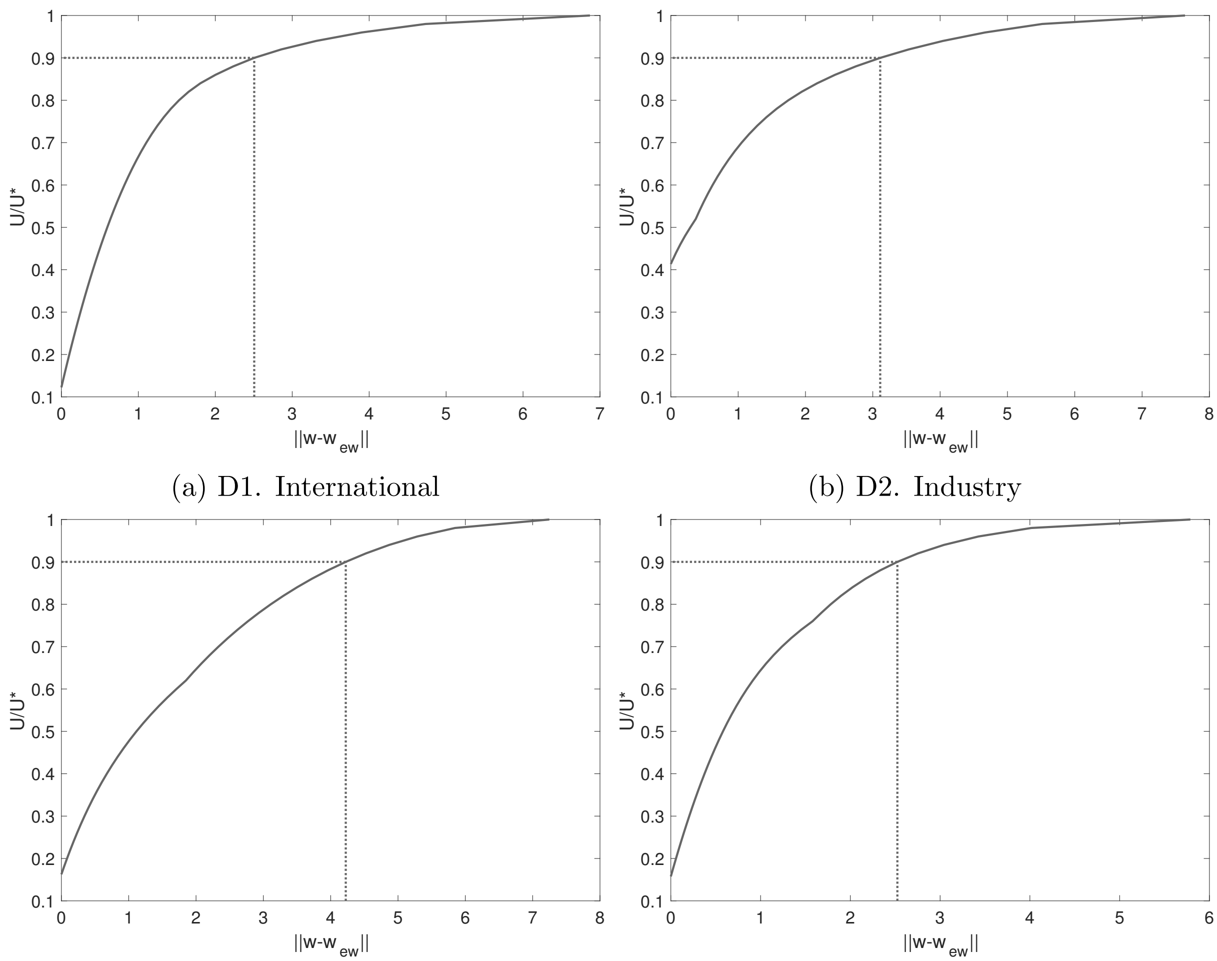
Turnover aversion is irrational behavior and results in a biased portfolio that allocates an excessive weight to the reference portfolio. However, the turnover aversion utility maximizing portfolio is less volatile and outperforms the usual utility maximizing portfolios in the presence of parameter uncertainty. Turnover aversion can also be incorporated into existing shrinkage estimators such as Kan and Zhou (2007) and Tu and Zhou (2011) to enhance their performance. These models aim to minimize the expected utility loss. However, with many assumptions and model parameter estimation involved, their actual performance is often unsatisfactory. As revealed by the simulation and empirical studies, their performance can be substantially improved when augmented with turnover aversion: the expected utility loss is minimized when these portfolios are shrunk further towards the reference portfolio. The degree of turnover aversion required for the minimum utility loss can be unexpectedly high, especially under a significant level of uncertainty.

Accounting for transaction costs in portfolio optimization is known to enhance robustness and performance, and has been endorsed by several authors: *e.g.*, Gârleanu and Pedersen (2013); DeMiguel et al. (2015); Olivares-Nadal and DeMiguel (2015). On the contrary, I find that using the equal-weight portfolio as the reference portfolio is far more effective than using the current portfolio, which is similar to accounting for transaction costs. This holds true even in the presence of transaction costs: the gap between the two versions indeed widens. The former renders less volatile portfolios and produces considerably better, more robust performance. Counter-intuitively, it also incurs lower transaction costs. This is a sharp contrast to the conventional wisdom: when the parameter uncertainty is significant, the current portfolio can be substantially different from the optimal portfolio, and penalizing the deviation from it does not guarantee better performance nor lower transaction costs compared to penalizing the deviation from the equal-weight portfolio.

Shrinking towards a reference portfolio does not necessarily involve a considerable utility loss as illustrated in the following example. Using the four datasets used in the simulation studies, the utility of the *ex-post* (true) optimal portfolio, U^*, and the utility of the equal-weight portfolio, U_{ew}, are calculated. Given a utility between U_{ew} and U^*, the portfolio with

[2]Unlike the usual definition of turnover which refers to the change from the current portfolio, turnover in this paper is defined more broadly as the change from any portfolio known at the time of rebalancing.

the minimum distance (in the sense of the Euclidean norm) from the equal-weight portfolio is then obtained via optimization. The curves in Figure 1 are constructed by connecting these portfolios. In each chart, the horizontal axis represents the distance from the equal-weight portfolio, and the vertical axis represents utility normalized by U^*. A quick visual inspection reveals that the distance between the equal-weight portfolio and the optimal portfolio can generally be reduced by more than half for only 10% loss of utility. This implies that a robust portfolio can be constructed by shrinking towards the equal-weight portfolio while sacrificing only a small fraction of utility.



(a) D1. International

(b) D2. Industry

(c) D5. Fama-French

(d) D8. Momentum

Figure 1: Minimum Distance Portfolios

This figure demonstrates the relationship between the robustness and utility loss of a portfolio. In each chart, the solid line represents the portfolio with the minimum distance from the equal-weight portfolio given a level of utility. The horizontal axis represents the distance from the equal-weight portfolio, and the vertical axis represents the utility normalized by the utility of the true optimal portfolio.

To determine the optimal degree of turnover aversion, I propose a data-driven calibration method. This method proves to be effective and enhance portfolio performance significantly when applied to various models and estimation window sizes. A fair and efficient portfolio evaluation method that aggregates the results from several datasets is also offered. This method facilitate performance evaluation when the test involves many datasets and performance measures. Both simulation and empirical studies show that the turnover aversion models perform superior compared to various existing models. They survive comprehensive robustness tests that involve different datasets, optimization criteria, and sample periods.

2 Optimal Portfolio under Turnover Aversion

2.1 Utility Maximization

It is assumed that the investor maximizes a quadratic utility of the form

$$\max_{w} U(w) = w'\mu - \frac{\gamma}{2} w'\Sigma w - \frac{\delta}{2}(w - w_0)'G(w - w_0), \tag{1}$$

where $\mu \in \mathbb{R}^N$ and $\Sigma \in \mathbb{R}^{N\times N}$ are the mean and covariance matrix of N asset returns in excess of the risk-free rate, $w \in \mathbb{R}^N$ is the portfolio weights, and γ is the risk aversion coefficient of the investor.[3] The last term on the right hand side represents the investor's aversion to turnover from a reference portfolio w_0, where δ is the turnover aversion coefficient and $G \in \mathbb{R}^{N\times N}$ is a penalty matrix. The reference portfolio w_0 can be any portfolio known at the time of portfolio rebalancing: the equal-weight portfolio, w_{ew}, and the current portfolio, w_{t-}, are considered in this paper. While the turnover aversion term looks similar to quadratic transaction costs when $w_0 = w_{t-}$ (*e.g.*, Gârleanu and Pedersen, 2013; Olivares-Nadal and DeMiguel, 2015), it has no association with transaction costs and is better interpreted as the investor's aversion to turnover or "psychological transaction costs" arising from parameter uncertainty: when the investor is not confident of the input parameter estimates, she would be reluctant to invest in the optimal portfolio obtained from those estimates.

The optimal portfolio w^* that maximizes the utility in (1) is given by[4]

$$w^* = (\gamma\Sigma + \delta G)^{-1}(\mu + \delta G w_0). \tag{2}$$

If an asset return has a large variance, its mean estimate may well have a large estimation error, and it is justifiable to penalize the turnover of that asset more severely. From this perspective, a natural choice of G would be the covariance matrix, Σ. When $G \equiv \Sigma$, the

[3]Returns refer to excess returns throughout the paper unless otherwise noted.

[4]w^* is used as a generic notation to denote any optimal portfolio throughout the paper.

optimal portfolio becomes a convex combination of the Markowitz (1952) optimal portfolio, $w_{ml} = \frac{1}{\gamma}\Sigma^{-1}\mu$, and the reference portfolio, w_0:

$$w^* = \frac{\gamma}{\gamma+\delta} w_{ml} + \frac{\delta}{\gamma+\delta} w_0. \tag{3}$$

To implement w^*, unknown μ and Σ need to be estimated. If the asset returns are *i.i.d.* normal random variables, the maximum likelihood (ML) estimates of μ and Σ, $\hat{\mu}$ and $\hat{\Sigma}$, are independent of each other and have the following distributions:

$$\hat{\mu} \sim \mathcal{N}\left(\mu, \frac{\Sigma}{T}\right), \quad \hat{\Sigma} \sim \mathcal{W}_N(T-1, \Sigma)\frac{1}{T}, \tag{4}$$

where T is the estimation window size, and $\mathcal{N}$ and $\mathcal{W}_N$ respectively denote the normal and Wishart distribution. To allow the case when asset returns are not *i.i.d.* or $\hat{\mu}$ and $\hat{\Sigma}$ are estimated separately, *e.g.*, using different sample sizes, a slightly relaxed assumption,

$$\hat{\mu} \sim \mathcal{N}\left(\mu, \frac{\Sigma}{K}\right), \quad \hat{\Sigma} \sim \mathcal{W}_N(T-1, \Sigma)\frac{1}{T}, \tag{5}$$

for some constant K, is made.

An unbiased estimate of the Markowitz portfolio is then given by

$$\hat{w}_{ml} = \frac{1}{\gamma}\tilde{\Sigma}^{-1}\hat{\mu}, \quad \tilde{\Sigma} = \frac{T}{T-N-2}\hat{\Sigma}. \tag{6}$$

However, as shown by Kan and Zhou (2007), plugging $\hat{w}_{ml}$ in (3) is not optimal in terms of the expected utility loss. Therefore, I consider the following portfolio strategy

$$w(a,b) = a\hat{w}_{ml} + bw_0, \tag{7}$$

and find a and b so that the expected utility loss is minimized, or equivalently, the expected out-of-sample utility (expected utility, henceforth) is maximized:

$$\max_{a,b} E[U(a,b)] = E\left[w(a,b)\mu - \frac{\gamma}{2}w(a,b)'\Sigma w(a,b) - \frac{\delta}{2}(w(a,b)-w_0)'\Sigma(w(a,b)-w_0)\right]. \tag{8}$$

Proposition 1. *The optimal a and b that maximizes the expected utility in* (8) *are given by*

$$a^* = \frac{\gamma}{\gamma+\delta} a_0^*, \tag{9}$$

$$b^* = \frac{\gamma}{\gamma+\delta} b_0^* + \frac{\delta}{\gamma+\delta}, \tag{10}$$

where

$$a_0^* = \frac{\theta^2 - \psi^2}{c_1\left(\frac{N}{K} + \theta^2\right) - \psi^2}, \tag{11}$$

$$b_0^* = \frac{c_1\left(\frac{N}{K} + \theta^2\right) - \theta^2}{c_1\left(\frac{N}{K} + \theta^2\right) - \psi^2} \frac{1}{\gamma} \frac{w_0'\mu}{w_0'\Sigma w_0}, \tag{12}$$

$$\theta^2 = \mu'\Sigma^{-1}\mu, \quad \psi^2 = \mu_0'\Sigma^{-1}\mu, \quad \mu_0 = \frac{w_0'\mu}{w_0'\Sigma w_0}\Sigma w_0, \tag{13}$$

and

$$c_1 = \frac{(T-2)(T-N-2)}{(T-N-1)(T-N-4)}. \tag{14}$$

See Appendix A.1 for proof. a_0^* and b_0^* are the optimal a and b when $\delta = 0$. The optimal portfolio is given by

$$\begin{aligned} w(a^*, b^*) &= a^* \hat{w}_{ml} + b^* w_0 \\ &= \frac{\gamma}{\gamma+\delta}\left(a_0^* \hat{w}_{ml} + b_0^* w_0\right) + \frac{\delta}{\gamma+\delta} w_0 \\ &= \frac{\gamma}{\gamma+\delta}\left(a_0^* \hat{w}_{ml} + (1-a_0^*) w_{im}\right) + \frac{\delta}{\gamma+\delta} w_0, \end{aligned} \tag{15}$$

where

$$w_{im} = \frac{1}{\gamma}\frac{w_0'\mu}{w_0'\Sigma w_0} w_0 = \frac{1}{\gamma}\Sigma^{-1}\mu_0. \tag{16}$$

Since $w_0'\mu_0 = w_0'\mu$ and w_{im} is proportional to w_0, w_{im} can be interpreted as the Markowitz portfolio when the mean returns are μ_0. Estimation of a^* and b^* are provided in Appendix A.2.

When $w_0 = w_{ew}$ and $\delta = 0$, this model becomes similar to the model of Tu and Zhou (2011) except they set $b = 1 - a$. There is no reason to assume $b = 1 - a$ apart from the obvious advantage of having less parameters. Furthermore, under this restriction, the proportion of $\hat{w}_{ml}$ to w_0 is no longer invariant to γ.

Viewed as a function of δ, *i.e.*, $w^*(\delta) = w(a^*, b^*|\delta)$, the optimal portfolio can be rewritten as

$$w^*(\delta) = \frac{\gamma}{\gamma+\delta} w^* + \frac{\delta}{\gamma+\delta} w_0, \tag{17}$$

where $w^* = a_0^* \hat{w}_{ml} + b_0^* w_0$ is the solution to the usual expected utility maximization problem without the turnover aversion term. In fact, any shrinkage estimator of the form, $w(a, b) = a\hat{w} + bw_0$ for some portfolio $\hat{w}$, has the optimal solution given in (17) with $w^* = a_0^* \hat{w} + b_0^* w_0$ being the optimal solution when $\delta = 0$ (a_0^* and b_0^* here are generic notations to denote the optimal values and not as defined in (11) and (12)).

2.2 Variance Minimization

Turnover aversion can also be incorporated into a variance minimization problem:

$$\begin{aligned} \min_w \; V(w) &= \frac{1}{2} w' \Sigma w + \frac{\delta}{2} (w - w_0)' \Sigma (w - w_0) \\ \text{subject to } & w' 1_N = 1, \end{aligned} \tag{18}$$

where $1_N \in \mathbb{R}^N$ is a vector of ones, and $w_0' 1_N = 1$ is assumed. The optimal portfolio that solves this is given by

$$w^* = \frac{1}{1+\delta} w_{mv} + \frac{\delta}{1+\delta} w_0, \tag{19}$$

where $w_{mv} = \frac{\Sigma^{-1} 1_N}{1_N' \Sigma^{-1} 1_N}$ is the global minimum-variance portfolio. It can be estimated unbiasedly from

$$\hat{w}_{mv} = \frac{\hat{\Sigma}^{-1} 1_N}{1_N' \hat{\Sigma}^{-1} 1_N}. \tag{20}$$

As before, I consider the following portfolio strategy

$$w(a) = a\hat{w}_{mv} + (1-a) w_0, \tag{21}$$

and find a so that the expected variance is minimized:

$$\min_a E[V(a)] = E\left[\frac{1}{2} w(a)' \Sigma w(a) + \frac{\delta}{2} (w(a) - w_0)' \Sigma (w(a) - w_0) \right]. \tag{22}$$

Since $\hat{w}_{mv}' 1_N = 1$ and $w_0' 1_N = 1$, the budget constraint is implicitly satisfied without further restriction.

Proposition 2. *The optimal a that minimizes the expected variance in* (22) *is given by*

$$a^* = \frac{1}{1+\delta} \frac{\sigma_0^2 - \sigma_{mv}^2}{\sigma_0^2 - \left(1 - \frac{N-3}{T-N+1}\right)\sigma_{mv}^2}, \tag{23}$$

where $\sigma_0^2 = w_0'\Sigma w_0$ and $\sigma_{mv}^2 = w_{mv}'\Sigma w_{mv} = (1_N'\Sigma^{-1}1_N)^{-1}$ are the variances of w_0 and w_{mv}, respectively.

See Appendix B for proof and estimation of a^*.

2.3 Optimal Portfolio Choice under Constraints

One drawback of the proposed portfolio models is that it is difficult to extend them to a constrained optimization problem, *e.g.*, utility maximization with short-sale constraints. This is also true for other models that maximize expected utility such as Kan and Zhou (2007) and Tu and Zhou (2011). To account for parameter uncertainty in a constrained problem, I adopt the following method. Let $\hat{w}^*$ denote the optimal portfolio that maximizes expected utility without constraints. The expected return implied by $\hat{w}^*$ can be derived from

$$\bar{\mu} = \gamma\hat{\Sigma}\hat{w}^*. \tag{24}$$

The constrained problem is then solved as usual after substituting $\hat{\mu}$ with $\bar{\mu}$.

Even though this method does not maximize the expected utility subject to the constraints, it turns out to be effective in the empirical studies. The same approach is adopted for a constrained variance minimization problem.

3 Data and Portfolio Models

3.1 The Data

The turnover aversion models are tested on the thirteen datasets described in Table 1 and compared against the portfolio models listed in Table 3. The datasets are based on the datasets used in DeMiguel et al. (2009), Kirby and Ostdiek (2012), and Kan et al. (2016), but also include new ones. Except for the first dataset D1 which has the sample period from 1990.10 to 2015.12, all other datasets have the same sample period from 1951.01 to 2015.12. The sample period refers to the out-of-sample period during which portfolios are rebalanced and evaluated, and the samples for moments estimation extend further to the past. For example, when T = 240, the mean and covariance matrix of the asset returns in the first month are estimated using the sample from 1931.01 to 1950.12. By using the same out-of-sample period regardless of the estimation window size, the results from different estimation window sizes ($T = 60$, 120, and 240 months in this paper) can be directly compared. The moments of the asset returns are updated monthly during the sample period rolling the estimation window.

Table 1: The Datasets

This table lists the datasets used in the simulation and empirical studies. The 8 international indices in D1 are the gross returns on large/mid cap stocks from eight countries: Canada, France, Germany, Italy, Japan, Switzerland, United Kingdom, and USA. The 20 portfolios with size-sort (D5, 6, 7, 11, 12, 20) are from the corresponding 25 portfolios excluding the 5 largest portfolios. All data are from K. French website (`http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html`) except D1, which is from the MSCI website (`https://www.msci.com/end-of-day-data-country`).

Dataset	Description	N	Sample Period
D1	8 International + World Indices	9	1990.10 - 2015.12
D2	10 Industry Portfolios + Market	11	1951.01 - 2015.12
D3	30 Industry Portfolios + Market	31	1951.01 - 2015.12
D4	3 Fama-French (FF) Factors	3	1951.01 - 2015.12
D5	20 FF Portfolios + Market	21	1951.01 - 2015.12
D6	20 FF Portfolios + FF 3	23	1951.01 - 2015.12
D7	20 FF Portfolios + FF 3 and Momentum	24	1951.01 - 2015.12
D8	10 Momentum Portfolios + Market	11	1951.01 - 2015.12
D9	10 Short-Term Reversal Portfolios + Market	11	1951.01 - 2015.12
D10	10 Long-Term Reversal Portfolios + Market	11	1951.01 - 2015.12
D11	20 Size/Momentum Portfolios + Market	21	1951.01 - 2015.12
D12	20 Size/Short-Term Reversal Portfolios + Market	21	1951.01 - 2015.12
D13	20 Size/Long-Term Reversal Portfolios + Market	21	1951.01 - 2015.12

Before assessing the performance of the portfolio models, it is worth understanding the characteristics of the datasets. Table 2 reports a summary of the *ex-post* optimal portfolios, *i.e.*, Markowitz portfolios obtained from the mean and covariance matrix of the entire sample. As evidenced from the sum of the absolute values of the weights (the fifth column), the *ex-post* optimal portfolios are unrealistically highly leveraged in most datasets. They even have a short position on the risky portfolio in D6 and D7. As shown in the last three columns, these datasets also frequently experience negative expected returns on the global minimum-variance portfolio, which will lead to a short risky portfolio position in the optimal portfolio.[5] High leverage arises largely from the inclusion of the market or factor portfolios: these portfolios can be approximated by the other assets and the optimal portfolio often has the form of a long-short strategy (sell the market and buy the other assets). Without

[5] There is a marked contrast between D5 which contains only the market portfolio and D6 and D7 which contain all three Fama-French factors: adding the Fama-French factors results in higher leverage and short positions on the risky portfolio.

the factor portfolios, the datasets behave more nicely resulting in less leveraged portfolios. Nevertheless, the empirical studies of this paper are primarily based on the datasets including factor portfolios as these have been used in many previous studies, *e.g.*, DeMiguel et al. (2009). The results from the datasets without factor portfolios are provided in Internet Appendix.

Table 2: *Ex-post* Optimal Portfolio Weights

This table summarizes the *ex-post* optimal portfolio weights from each dataset. The *ex-post* optimal portfolio is defined as the Markowitz portfolio obtained from the sample moments of the entire sample. '$\min w_i$' and '$\max w_i$' are respectively the minimum and maximum weights on the risky assets, '$\sum w_i$' is the sum of the risky asset weights, *i.e.*, the weight of the risky portfolio, and '$\sum |w_i|$', the sum of the absolute values of the weights, measures the degree of leverage. The last three columns are the frequency of negative expected returns on the global minimum-variance portfolio during the sample period for the estimation window size $T = 60$, 120, and 240.

	$\min w_i$	$\max w_i$	$\sum w_i$	$\sum \lvert w_i \rvert$	$P(\mu_g < 0)$		
					60	120	240
D1	-5.37	3.87	1.40	12.20	0.28	0.13	0.00
D2	-6.91	1.79	1.89	15.71	0.09	0.02	0.00
D3	-7.03	1.55	1.92	21.43	0.23	0.13	0.00
D4	0.53	2.12	3.98	3.98	0.10	0.02	0.00
D5	-2.32	3.54	2.34	26.74	0.22	0.09	0.00
D6	-4.17	3.89	-4.19	38.56	0.63	0.74	0.87
D7	-5.75	3.43	-3.64	41.90	0.66	0.75	0.88
D8	-4.02	2.47	1.53	14.40	0.17	0.11	0.06
D9	-1.42	1.45	1.51	9.64	0.23	0.19	0.12
D10	-3.55	1.31	1.46	10.67	0.15	0.08	0.00
D11	-4.23	3.79	2.39	25.21	0.11	0.00	0.00
D12	-7.12	3.65	1.24	31.71	0.20	0.22	0.15
D13	-2.65	2.37	2.21	18.79	0.14	0.01	0.00

3.2 The Portfolio Models

Table 3 lists the portfolio models that are compared in this paper. The *ex-post* optimal portfolio (W*) is the Markowitz portfolio obtained from the sample moments of the entire sample. The equal-weight portfolio (EW) is chosen as a benchmark, and other standard portfolio strategies, *i.e.*, the Markowitz optimal portfolio (ML), global minimum-variance portfolio (MV), and their short-sale constrained versions (ML+, MV+) are also considered. The models (VT, OC) of Kirby and Ostdiek (2012) are added as they are argued to outperform EW. The turnover minimization models (TML, TMV) of Han (2016), which seek a sub-optimal portfolio with a minimum distance from a reference portfolio, are also included. The three-fund rule (KZ) of Kan and Zhou (2007) and the shrinkage estimators (TZML, TZKZ) of Tu and Zhou (2011) are included as they share the same approach to address parameter uncertainty and are similar to the proposed models when the turnover aversion term is absent ($\delta = 0$).

The turnover aversion models (TAML, TAMV) are tested using different degrees of turnover aversion and two reference portfolios, w_{ew} and w_{t-}.[6] In addition, a model (TAMLK) that estimates K in (5) instead of assuming $K = T$ is examined. The estimation method is described in Appendix A.3.

Variants of KZ, TZML, and TZKZ that incorporate turnover aversion are also considered. When turnover aversion is incorporated into TZML or TZKZ, the optimal portfolio is given by

$$w_{tz}(\delta) = \frac{\gamma}{\gamma + \delta} w_{tz} + \frac{\delta}{\gamma + \delta} w_0, \tag{25}$$

where w_{tz} is the original Tu and Zhou portfolio (TZML or TZKZ).[7] Extension of KZ is less straightforward. In principle, it would be best to determine the coefficients on the three portfolios simultaneously by letting

$$w(a, b, c) = a\hat{w}_{ml} + b\hat{w}_{mv} + cw_0, \tag{26}$$

and determining a, b, and c so that the expected utility is maximized. The solution to this problem is given in Appendix C. While theoretically superior, this formulation is difficult to implement due to the complexity of model parameter estimation: simulations show that a crude plug-in method using the ML estimates, $\hat{\mu}$ and $\hat{\Sigma}$, performs unsatisfactorily. Instead, using the generic solution in (17), the following form is employed:

$$w_{kz}(\delta) = \frac{\gamma}{\gamma + \delta} w_{kz} + \frac{\delta}{\gamma + \delta} w_0, \tag{27}$$

where w_{kz} is the original Kan and Zhou three-fund rule.

Implementation details of each model can be found in Internet Appendix.

4 Simulation Studies

The turnover aversion models are first validated via simulation studies. Four datasets, D1, D2, D5, and D8, out of the thirteen datasets in Table 1 are chosen for simulation. The sample mean and covariance matrix of the entire sample are regarded as the true mean and

[6] More precisely, the portfolio weights of the previous month, w_{t-1}, is used instead of w_{t-} which reflects the return over the past month. This is because w_{t-} can have abnormal values when the portfolio is highly leveraged and as a consequence influence the new portfolio adversely.

[7] Strictly speaking, Equation (25) holds only when $w_0 = w_{ew}$, but is used also when $w_0 = w_{t-}$.

Table 3: The Portfolio Models

This table lists the portfolio models considered in the simulation and empirical studies. The models with '+' in their abbreviation are those subject to the short-sale constraint. The short-sale constraint is applied only to risky assets. Implementation details of each model can be found in Internet Appendix.

Abbreviation	Description
W*	*Ex-post* optimal portfolio, *i.e.*, the Markowitz portfolio obtained from the sample moments of the entire sample.
EW	Equal-weight portfolio.
Standard Portfolio Strategies	
ML, ML+	Standard Markowitz (1952) mean-variance optimal portfolio.
MV, MV+	Global minimum-variance portfolio.
Kirby and Ostdiek (2012)	
VT	Volatility timing strategy.
OC, OC+	Optimal constrained portfolio: the Markowitz portfolio without the risk-free asset.
Han (2016)	
TML(τ), TML+(τ)	Utility maximization-Turnover minimization. $w_0 = w_{ew}$.
TMLc(τ), TMLc+(τ)	Utility maximization-Turnover minimization. $w_0 = w_{t-}$.
TMV(τ), TMV+(τ)	Variance minimization-Turnover minimization. $w_0 = w_{ew}$.
TMVc(τ), TMVc+(τ)	Variance minimization-Turnover minimization. $w_0 = w_{t-}$.
τ: tolerance factor	
Kan and Zhou (2007)	
KZ	Kan and Zhou (2007) three-fund rule.
KZ(δ), KZc(δ)	KZ with turnover aversion. KZ(δ): $w_0 = w_{ew}$; KZc(δ): $w_0 = w_{t-}$.
Tu and Zhou (2011)	
TZML	Tu and Zhou (2011) model that combines ML with EW.
TZML(δ), TZMLc(δ)	TZML with turnover aversion. TZML(δ): $w_0 = w_{ew}$; TZMLc(δ): $w_0 = w_{t-}$.
TZKZ	Tu and Zhou (2011) model that combines KZ with EW.
TZKZ(δ), TZKZc(δ)	TZKZ with turnover aversion. TZKZ(δ): $w_0 = w_{ew}$; TZKZc(δ): $w_0 = w_{t-}$.
Turnover Aversion Models	
TAML(δ), TAML+(δ)	Utility maximization with turnover aversion. $w_0 = w_{ew}$.
TAMLc(δ), TAMLc+(δ)	Utility maximization with turnover aversion. $w_0 = w_{t-}$.
TAMLK(δ)	TAML(δ) with estimated K.
TAMV(δ), TAMV+(δ)	Variance minimization with turnover aversion. $w_0 = w_{ew}$.
TAMVc(δ), TAMVc+(δ)	Variance minimization with turnover aversion. $w_0 = w_{t-}$.
δ: turnover aversion coefficient	

covariance matrix. The expected utility and variance are obtained from 10,000 iterations.[8] These values are computed without the turnover aversion term as *ex-post* performance should not be affected by turnover aversion.

4.1 Utility Maximization

Table 4 reports normalized expected utilities for the case of $\gamma = 3$. The first column represents the portfolio models, and the numbers in the header are estimation window sizes. The figures in the table are averages across the datasets. Detailed results from each dataset can be found in Internet Appendix.

Table 4: Expected Utility: $\gamma = 3$

This table reports the mean and standard error of the utilities of selected portfolios, obtained from 10,000 iterations. Utilities are normalized by that of W*. The reported values are the averages across the datasets; D1, D2, D5, and D8. The numbers in the header are estimation window sizes. EW and EW* are equal-weight portfolios adjusted to maximize utility, respectively using sample and true moments.

	Mean					Standard Error				
	60	120	240	360	600	60	120	240	360	600
W*	1.000	1.000	1.000	1.000	1.000	0.000	0.000	0.000	0.000	0.000
EW*	0.230	0.230	0.230	0.230	0.230	0.000	0.000	0.000	0.000	0.000
EW	-0.024	0.110	0.172	0.191	0.207	0.407	0.183	0.082	0.056	0.032
ML	-6.094	-1.074	0.167	0.480	0.705	4.850	1.094	0.387	0.233	0.128
KZ	0.058	0.379	0.587	0.681	0.782	0.654	0.295	0.162	0.123	0.086
TZML	0.183	0.395	0.574	0.671	0.778	0.530	0.247	0.159	0.127	0.090
TZKZ	0.259	0.436	0.581	0.664	0.764	0.363	0.183	0.135	0.117	0.091
TAML(0)	0.200	0.402	0.576	0.672	0.778	0.507	0.238	0.156	0.126	0.090
TAML(1)	0.286	0.431	0.571	0.653	0.747	0.274	0.161	0.138	0.121	0.092
TAML(2)	0.312	0.424	0.541	0.611	0.693	0.185	0.140	0.132	0.117	0.092
TAML(3)	0.318	0.409	0.509	0.570	0.642	0.145	0.128	0.124	0.111	0.088
ML+	-0.144	0.153	0.291	0.335	0.374	0.744	0.297	0.130	0.087	0.050
TAML+(0)	0.280	0.311	0.347	0.366	0.389	0.079	0.068	0.055	0.046	0.034
TAML+(1)	0.281	0.311	0.345	0.365	0.387	0.065	0.059	0.050	0.043	0.031

Consistent with the findings in the literature, *e.g.*, Tu and Zhou (2011) and Kan et al. (2016), ML is outperformed by EW when the window size is small: $T > 240$ is required for ML to outperform EW. KZ improves over ML significantly and outperforms both EW and ML across all window sizes. It is however outperformed by the two models of Tu and Zhou (2011). Of the two, TZKZ performs particularly well.

[8] For certain models, the expected utility can be obtained analytically: *e.g.*, Kan et al. (2016) derive an analytic formula for the expected utility of the Kan and Zhou (2007) three-fund rule. However, to calculate both the mean and standard error of the utility, simulation is employed for all models. By sampling directly from the distributions of $\hat{\mu}$ and $\hat{\Sigma}$ rather than the asset returns, simulation efficiency can be improved.

Comparing TAML(0) with TZML, TAML(0) is found to marginally, but consistently outperform TZML. It also has smaller standard errors. This result is in favour of the proposed two-parameter model over the one-parameter model of Tu and Zhou (2011). In fact, the difference becomes more prominent when $\gamma = 1$ (available in Internet Appendix).

Of particular interest is the effect of turnover aversion, which can be examined by comparing the TAML models with different values of the turnover aversion coefficient δ. Performance enhancement stemming from the inclusion of the turnover aversion term is substantial when T is small: the expected utility of TAML(0) is 0.200 when $T = 60$, whereas those of TAML(1) and TAML(2) are 0.286 and 0.312, respectively. These values are even higher than the expected utility of EW* which assumes prior knowledge of the distribution. This result is rather striking considering that TAML(δ), $\delta > 0$, is a linear combination of TAML(0) and EW*. Furthermore, the fact that incorporating turnover aversion causes the objective utility function to drift away from the evaluation utility function makes the result particularly remarkable. Due to this misalignment, the δ associated with the maximum utility declines as T increases, *i.e.*, as parameter uncertainty diminishes, and the turnover aversion models eventually underperform TAML(0). This result suggests that a carefully chosen δ for a given level of parameter uncertainty would improve the performance of the optimal portfolio. This hypothesis is further investigated later in this section.

Incorporating turnover aversion does not only enhance the expected utility but also reduces its standard error considerably: *e.g.*, when $T = 60$, the standard error of TAML(0) is 0.507, whilst those of TAML(1) and TAML(2) are only 0.274 and 0.185, respectively. Smaller standard error implies a smaller chance of extreme losses over a finite investment horizon.

Jagannathan and Ma (2003) show that imposing short-sale constraints can reduce estimation error even when the constraints are wrong. A similar conclusion can be drawn here: ML+, TAML+(0), and TAML+(1) all exhibit superior performance to ML when $T < 360$. This result is impressive considering the high leverage of the *ex-post* optimal portfolios as reported in Table 2. The performance of TAML+ is particularly noteworthy: it outperforms ML+ significantly and outperforms EW* across all T's. Besides, it has significantly lower standard errors compared to ML+. This suggests that the proposed method of incorporating constraints into the turnover aversion models is effective. Nevertheless, a limitation of short-sale constrained models is that their performance is suppressed even when T is large.

Unlike simulation, the real-world performance of optimal portfolios does not necessarily improve with the estimation window size: DeMiguel et al. (2009) find that accumulating the estimation window rather than rolling it improves the performance of optimal models only slightly,[9] whereas no apparent relationship between performance and window size can be

[9] B.2 of the online appendix.

derived from the empirical results in Tu and Zhou (2011). It appears that beyond a certain window size, parameter uncertainty does not diminish further or even rises again. From this perspective, the turnover aversion models that show robust performance when T is small are expected to demonstrate superior performance when applied to actual market data.

The same simulation is repeated with $\gamma = 1$ and the results are reported in Internet Appendix. The overall results are similar to those with $\gamma = 3$ except the followings. When $\gamma = 1$, the short-sale constrained models do not perform well anymore. The expected utilities of these models are only about a half of the expected utilities of the TAML models. This implies that the short-sale constraint could deprive less-risk-averse investors of opportunities to seek additional returns. Another exception is that the performance difference between TZML and TAML(0) becomes more prominent. This can be traced to the fact that the allocation between ML and EW in TZML is γ-dependent.

4.1.1 Sensitivity to Misspecification of Mean Return

The assumption that the returns are *i.i.d.* random variables is rather strong and unrealistic, and the expectation of the sample mean may well deviate from the true mean. To examine the effect of misspecification of the mean, it is perturbed when random samples are drawn by adding $0.2\text{diag}(\mu)z$ to μ in (5), where $\text{diag}(\mu)$ is a diagonal matrix with μ in its diagonal, and z is an N-dimensional standard normal random variable. Simulation results are reported in Table 5.

It is striking how small errors in mean can deteriorate the performance of shrinkage estimators: KZ, TZML, TZKZ, and TAML(0) all perform poorly and yield negative utilities regardless of the size of T. In fact, the performance of these models worsens with T. This is because these models put more weight on ML as T increases, ignoring the misspecification. This may explain to some extent why some shrinkage estimators perform worse when the estimation window size is larger: see, *e.g.*, Table 6 of Tu and Zhou (2011). On the contrary, the turnover aversion models are much more robust to misspecification. TAML with $\delta > 0$ maintains positive expected utility and its performance improves with T. The short-sale constrained models are also robust to the misspecification of the mean. Furthermore, the turnover aversion and short-sale constrained models have much smaller standard errors.

Accounting for parameter uncertainty does help improve portfolio performance. KZ, TZML, TZKZ, and TAML(0) all improve over ML and generally outperform EW even for a moderately large T. However, since their model parameters need to be estimated, these models are still sensitive to estimation errors and misspecification, and their actual performance could be unexpectedly poor. Meanwhile, the turnover aversion models are robust

Table 5: Expected Utility: $\gamma = 3$, Error in Mean

This table reports the mean and standard error of the utilities of selected portfolios, obtained from 10,000 iterations. To simulate misspecification of the mean, it is perturbed by adding $0.2\text{diag}(\mu)z$ to μ in (5), where $\text{diag}(\mu)$ is a diagonal matrix with μ in its diagonal, and z is an N-dimensional standard normal random variable. Utilities are normalized by that of W*. The reported values are the averages across the datasets; D1, D2, D5, and D8. The numbers in the header are estimation window sizes. EW and EW* are equal-weight portfolios adjusted to maximize utility, respectively using sample and true moments.

	Mean					Standard Error				
	60	120	240	360	600	60	120	240	360	600
W*	1.000	1.000	1.000	1.000	1.000	0.000	0.000	0.000	0.000	0.000
EW*	0.230	0.230	0.230	0.230	0.230	0.000	0.000	0.000	0.000	0.000
EW	-0.027	0.108	0.171	0.191	0.206	0.415	0.181	0.086	0.056	0.034
ML	-10.210	-3.321	-1.581	-1.139	-0.831	7.851	2.522	1.521	1.276	1.151
KZ	-0.473	-0.335	-0.315	-0.325	-0.352	1.370	1.057	0.988	0.955	0.971
TZML	-0.317	-0.302	-0.325	-0.342	-0.368	1.287	1.035	0.981	0.951	0.968
TZKZ	-0.027	0.014	-0.043	-0.097	-0.180	0.902	0.767	0.814	0.829	0.892
TAML(0)	-0.293	-0.291	-0.320	-0.338	-0.366	1.266	1.029	0.979	0.950	0.968
TAML(1)	0.039	0.091	0.121	0.133	0.141	0.681	0.558	0.541	0.532	0.546
TAML(2)	0.173	0.239	0.287	0.309	0.329	0.430	0.364	0.360	0.359	0.370
TAML(3)	0.235	0.303	0.357	0.382	0.406	0.305	0.270	0.272	0.274	0.282
ML+	-0.183	0.117	0.247	0.289	0.322	0.798	0.328	0.164	0.117	0.085
TAML+(0)	0.273	0.297	0.321	0.333	0.346	0.101	0.094	0.082	0.073	0.065
TAML+(1)	0.281	0.308	0.334	0.347	0.360	0.076	0.071	0.063	0.057	0.051

to misspecification and demonstrate superior performance when subject to large estimation errors. In addition, they have much smaller standard errors.

4.1.2 Performance over a Finite Investment Horizon

The results in Table 4 and 5 are asymptotic properties. A real-world investment horizon is finite and the performance of portfolios can be different. To examine the performance over a finite investment horizon, portfolios are assumed to be managed for ten years during which they are rebalanced monthly. As we now have the "current portfolio", the TAML models with $w_0 = w_{t-}$ are also examined.

Table 6 reports the mean and standard deviation of the normalized certainty equivalents (CE) obtained from 10,000 iterations. The results in the upper (lower) panel are before (after) transaction costs. Transaction costs of 10 basis points (bp) for both buying and selling risky assets and 0 bp for the risk-free asset are assumed. The mean certainty equivalents are similar to the expected utilities in Table 4 and will not be discussed further. What is more interesting is a comparison between the TAML models with $w_0 = w_{ew}$ (TAML(δ)) and those with $w_0 = w_{t-}$ (TAMLc(δ)). When $T > 60$, TAMLc(δ) has a higher CE than its counterpart. Adding the current portfolio has an effect similar to accumulating estimation

Table 6: Certainty Equivalent: $\gamma = 3$

This table reports the mean and standard error of the certainty equivalents (CE) of selected portfolios, obtained from 10,000 iterations. Portfolios are assumed to be rebalanced monthly and managed for ten years. CE's are normalized by that of W*. The reported values are the averages across the datasets; D1, D2, D5, and D8. The numbers in the header are estimation window sizes. EW and EW* are equal-weight portfolios adjusted to maximize utility, respectively using sample and true moments.

	Mean					Standard Error				
	60	120	240	360	600	60	120	240	360	600
W*	1.000	1.000	1.000	1.000	1.000	0.653	0.650	0.649	0.653	0.647
EW*	0.232	0.229	0.231	0.228	0.228	0.306	0.304	0.302	0.307	0.305
EW	-0.018	0.114	0.175	0.191	0.206	0.385	0.358	0.338	0.334	0.322
ML	-5.994	-1.021	0.193	0.494	0.715	1.778	1.024	0.873	0.812	0.748
KZ	0.064	0.388	0.596	0.685	0.785	0.545	0.559	0.577	0.591	0.597
TZML	0.188	0.400	0.581	0.675	0.781	0.478	0.505	0.546	0.569	0.586
TZKZ	0.261	0.439	0.586	0.666	0.765	0.449	0.469	0.495	0.515	0.533
TAML(0)	0.205	0.407	0.583	0.676	0.781	0.470	0.500	0.541	0.565	0.584
TAML(1)	0.290	0.434	0.575	0.654	0.746	0.431	0.455	0.477	0.489	0.494
TAML(2)	0.314	0.426	0.544	0.611	0.691	0.405	0.423	0.437	0.443	0.443
TAML(3)	0.320	0.410	0.511	0.570	0.640	0.387	0.400	0.409	0.413	0.411
ML+	-0.130	0.163	0.296	0.337	0.374	0.534	0.521	0.496	0.484	0.464
TAML+(0)	0.282	0.311	0.348	0.366	0.388	0.354	0.369	0.382	0.395	0.404
TAML+(1)	0.283	0.311	0.346	0.364	0.386	0.343	0.353	0.362	0.371	0.376
TAMLc(0)	0.126	0.446	0.596	0.680	0.781	0.804	0.589	0.561	0.571	0.585
TAMLc(1)	0.226	0.483	0.600	0.667	0.750	0.700	0.510	0.478	0.482	0.488
TAMLc(2)	0.272	0.490	0.585	0.639	0.707	0.650	0.475	0.440	0.441	0.442
TAMLc(3)	0.297	0.489	0.573	0.621	0.687	0.622	0.458	0.423	0.424	0.426

(a) Before Transaction Cost

	Mean					Standard Error				
	60	120	240	360	600	60	120	240	360	600
W*	1.000	1.000	1.000	1.000	1.000	0.711	0.707	0.706	0.711	0.704
EW*	0.251	0.247	0.250	0.247	0.247	0.333	0.331	0.328	0.334	0.332
EW	-0.035	0.117	0.187	0.205	0.222	0.418	0.388	0.367	0.363	0.350
ML	-8.284	-1.655	-0.053	0.343	0.631	2.186	1.138	0.948	0.882	0.812
KZ	-0.172	0.278	0.546	0.655	0.770	0.577	0.590	0.616	0.634	0.644
TZML	0.006	0.304	0.533	0.644	0.765	0.510	0.530	0.580	0.609	0.632
TZKZ	0.137	0.387	0.569	0.661	0.770	0.473	0.494	0.526	0.551	0.574
TAML(0)	0.030	0.315	0.536	0.645	0.765	0.500	0.525	0.575	0.605	0.629
TAML(1)	0.178	0.383	0.560	0.654	0.760	0.454	0.479	0.509	0.524	0.533
TAML(2)	0.236	0.395	0.542	0.623	0.715	0.428	0.448	0.467	0.477	0.479
TAML(3)	0.261	0.391	0.517	0.588	0.669	0.410	0.426	0.439	0.446	0.445
ML+	-0.185	0.154	0.309	0.357	0.400	0.578	0.564	0.539	0.525	0.504
TAML+(0)	0.288	0.326	0.370	0.392	0.417	0.383	0.399	0.415	0.429	0.439
TAML+(1)	0.291	0.326	0.369	0.390	0.415	0.372	0.383	0.393	0.403	0.409
TAMLc(0)	0.020	0.423	0.593	0.680	0.783	0.921	0.652	0.611	0.621	0.635
TAMLc(1)	0.152	0.480	0.617	0.689	0.776	0.787	0.556	0.517	0.522	0.528
TAMLc(2)	0.214	0.495	0.608	0.668	0.741	0.725	0.517	0.476	0.477	0.479
TAMLc(3)	0.248	0.498	0.599	0.652	0.722	0.691	0.497	0.457	0.459	0.462

(b) After Transaction Cost

sample and TAMLc(δ) is anticipated to outperform TAML(δ) under the *i.i.d.* assumption. Notwithstanding, it underperforms TAML(δ) when $T = 60$ both before and after transaction costs, and has a higher standard error. This suggests that when the moments are subject to large estimation errors, the current portfolio may well be far from the true optimal portfolio and anchoring the portfolio to the current portfolio can be less effective than anchoring it to a fixed-weight portfolio. As illustrated in the next section, the equal-weight portfolio indeed serves better as the reference portfolio.

4.1.3 Optimal Degree of Turnover Aversion

The results so far suggest that turnover aversion does improve the performance of optimal portfolios, and the optimal degree of turnover aversion decreases as estimation window size increases and therefore parameter uncertainty diminishes. This section investigates the optimal degree of turnover aversion for four models; KZ, TZML, TZKZ, and TAML. These models are similar in that they maximize the expected utility by optimally combining multiple portfolios.

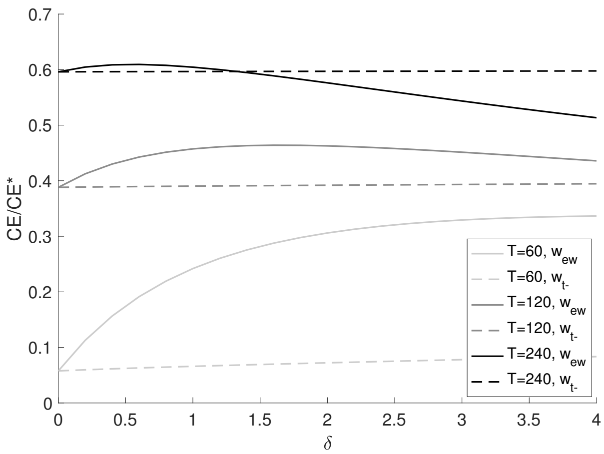
Figure 2 displays the certainty equivalent of each model as a function of δ in the absence of transaction costs. Solid lines are for $w_0 = w_{ew}$ and dashed lines are for $w_0 = w_{t-}$. Adding the turnover aversion term normally improves the performance of the models, and its effect is particularly noticeable when $w_0 = w_{ew}$ and T is small: *e.g.*, the CE of KZ increases by 20% at $\delta \approx 1.6$ when $T = 120$, while it continues to increase with δ within the considered range when $T = 60$. Among the four models, only KZ does not contain w_{ew} in its original form and thus benefits most by incorporating w_{ew}, whilst TZKZ which includes both w_{ew} and w_{kz} benefits least from turnover aversion. In contrast to the case of $w_0 = w_{ew}$, penalizing the turnover from w_{t-} has little effect on performance when there is no transaction cost.[10] This is because the expected portfolio weights of the turnover aversion models with $w_0 = w_{t-}$ converge to those of their base models, whilst their variances are not particularly smaller. The moments of the turnover aversion portfolio weights are established in the following proposition.

Proposition 3. *Suppose that a portfolio w is rebalanced t times via a turnover aversion model. Let w_t^c and w_t^e respectively denote the optimal portfolio at time t when $w_0 = w_{t-}$ and $w_0 = w_{ew}$:*

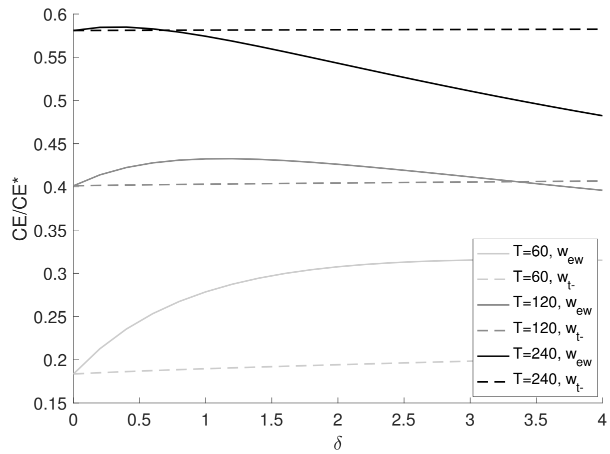
$$\begin{aligned} w_t^c &= (1-\alpha)w_t^* + \alpha w_{t-1}^c, \\ w_t^e &= (1-\alpha)w_t^* + \alpha w_{ew}, \end{aligned} \tag{28}$$

where w_t^* is the optimal portfolio of the base model, and $\alpha = \dfrac{\delta}{\gamma + \delta}$. The first and second

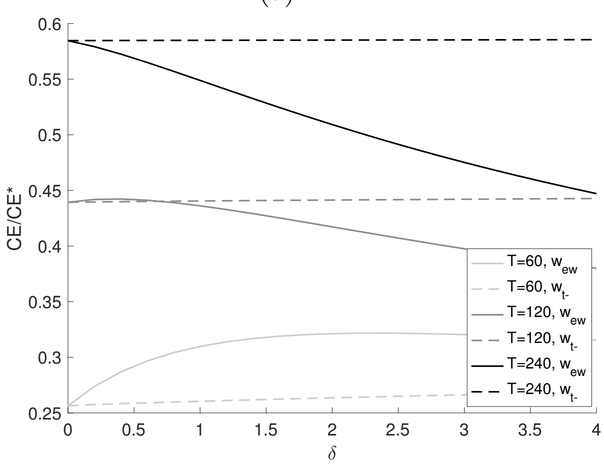
[10] An exception to this is TAML, which is discussed further later in this section.



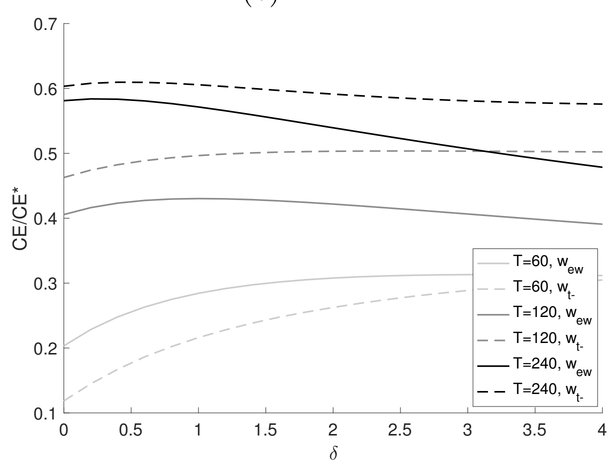
(a) KZ



(b) TZML



(c) TZKZ



(d) TAML

Figure 2: Optimal Turnover Aversion: No Transaction Costs

This figure displays the certainty equivalents of four portfolio strategies for different values of δ and estimation window sizes. Portfolios are assumed to be rebalanced monthly and managed for ten years. The vertical axis represents the normalized certainty equivalent averaged across the datasets; D1, D2, D5, and D8.

moments of the portfolio weights are given by:

$$\begin{aligned} E(w_t^c) &= (1-\alpha^t)E(w_t^*) + \alpha^t w, \\ V(w_{it}^c) &= (1-\alpha)^2 V(w_{it}^* + \alpha w_{it-1}^* + \cdots + \alpha^{t-1} w_{i1}^*), \\ E(w_t^e) &= (1-\alpha)E(w_t^*) + \alpha w_{ew}, \\ V(w_{it}^e) &= (1-\alpha)^2 V(w_{it}^*), \end{aligned} \tag{29}$$

where w_{it}^c and w_{it}^e are the i-th element of w_t^c and w_t^e, respectively. It follows that

$$E(w_t^c) \to E(w_t^*) \text{ as } t \to \infty, \tag{30}$$

$$V(w_{it}^e) < V(w_{it}^c) < V(w_{it}^*). \tag{31}$$

See Appendix D.1 for proof. Although w_t^e is biased, $V(w_{it}^e)$ can be considerably smaller than $V(w_{it}^*)$, resulting in better performance especially under high parameter uncertainty. On the other hand, while w_t^c is unbiased, $V(w_{it}^c)$ can be close to $V(w_{it}^*)$ especially when T is large, and the performance of w_t^c can be similar to that of w_t^*.

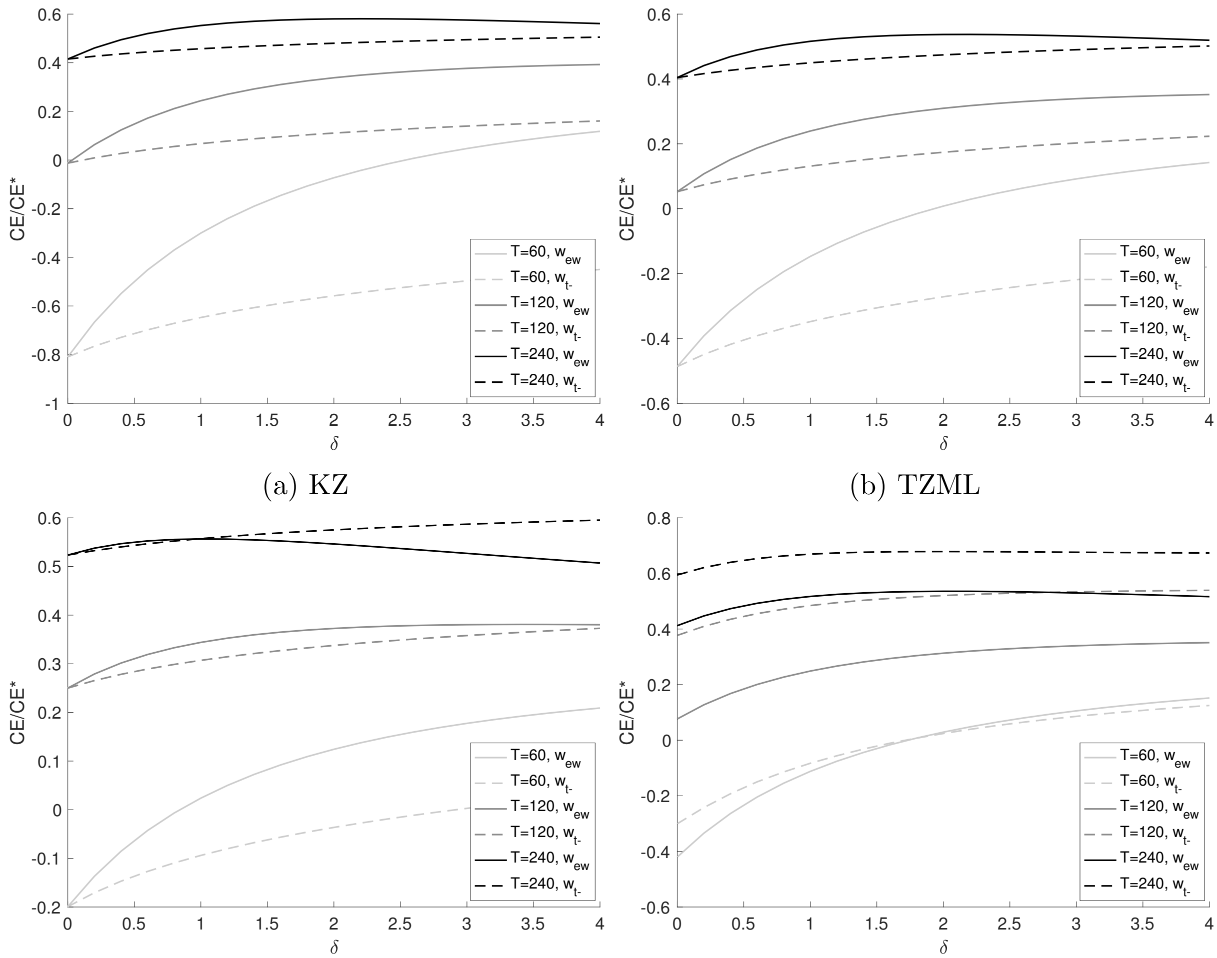
The effect of turnover aversion is more pronounced when transaction costs are taken into account. The results in Figure 3 are obtained assuming transaction costs of 30 bp.[11] In the presence of transaction costs, the models with $w_0 = w_{ew}$ as well as those with $w_0 = w_{t-}$ improve portfolio performance, but the improvement is more prominent when $w_0 = w_{ew}$. The optimal δ that maximizes CE is unexpectedly large often exceeding the range considered: recall that, with $\gamma = 3$, $\delta = 3$ implies 50 percent on the reference portfolio. When $w_0 = w_{ew}$, the actual loading on w_{ew} is even higher than what δ implies as TZML, TZKZ, and TAML already involve w_{ew} without turnover aversion.

It is striking that anchoring to w_{ew} is more effective than anchoring to w_{t-} even in the presence of transaction costs: the gap between the two versions indeed widens with transaction costs. This is because, as opposed to our intuition, using w_{ew} usually incurs lower transaction costs. To compare the transaction costs associated with the turnover aversion models, the following proposition establishes the relationship between the expected squared turnovers.[12]

Proposition 4. *Let $\Delta w_{it} = w_{it} - w_{it-1}$. If $E\left[\Delta w_{it}^* \Delta w_{it-1}^c\right] > -\frac{\alpha}{2(1-\alpha)} E\left[(\Delta w_{it-1}^c)^2\right]$, the*

[11] 10 bp and 50 bp were also tested and the overall pictures were similar to the case of 30 bp.

[12] In the proposition, turnover is defined as $w_{it} - w_{it-1}$ instead of $w_{it} - w_{it-} = w_{it} - (1 + r_{it-1})w_{it-1}$, for simplicity. In the latter case, the inequalities hold under a slightly more complex assumption.



(a) KZ (b) TZML

(c) TZKZ (d) TAML

Figure 3: Optimal Turnover Aversion: 30 bp Transaction Costs

This figure displays the certainty equivalents of four portfolio strategies for different values of δ and estimation window sizes in the presence of 30 bp transaction costs. Portfolios are assumed to be rebalanced monthly and managed for ten years. The vertical axis represents the normalized certainty equivalent averaged across the datasets; D1, D2, D5, and D8.

following inequalities hold:

$$E\left[(\Delta w_{it}^e)^2\right] < E\left[(\Delta w_{it}^c)^2\right] < E\left[(\Delta w_{it}^*)^2\right]. \tag{32}$$

See Appendix D.2 for proof. $E\left[\Delta w_{it}^* \Delta w_{it-1}^c\right] > -\frac{\alpha}{2(1-\alpha)} E\left[(\Delta w_{it-1}^c)^2\right]$ is a reasonable assumption as the correlation between Δw_{it}^* and Δw_{it-1}^c is usually small and negligible when the estimation window is large. If they are uncorrelated, the following relationship can be established.

Corollary 1. *If* $E\left[\Delta w_{it}^* \Delta w_{it-1}^c\right] = 0$,

$$E\left[(\Delta w_{it}^e)^2\right] < (1-\alpha^2) E\left[(\Delta w_{it}^c)^2\right]. \tag{33}$$

Proof follows immediately from $E\left[(\Delta w_{it-1}^c)^2\right] < E\left[(\Delta w_{it}^c)^2\right]$. The proposition suggests that both turnover aversion models reduce transaction costs, but incorporating w_{ew} is more effective especially when α is large.

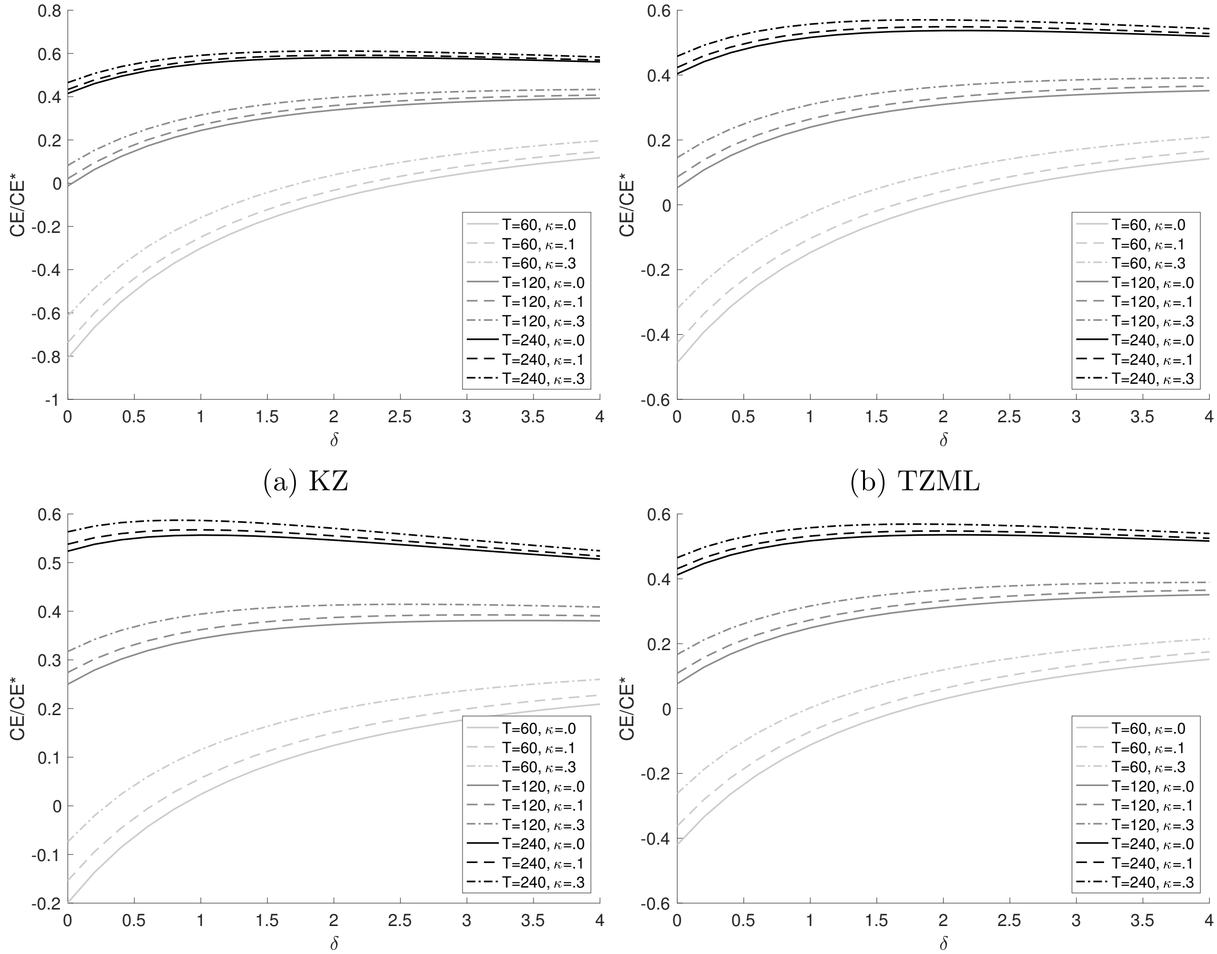
TAMLc is different from the other turnover aversion models with $w_0 = w_{t-}$ in that it consists only of $\hat{w}_{ml}$ and w_{t-} and their loadings are dynamically determined based on the input parameters, whereas the other models consist of the base model and w_{t-} and their loadings are solely determined by δ. This feature leads to the superior performance of TAMLc to TAML when the *i.i.d.* assumption holds and the estimation window is large. However, as revealed in the case of $T = 60$, TAMLc can be outperformed by TAML when uncertainty is high. In addition, TAML performance improves faster with δ. Since $\hat{w}_{ml}$, w_{t-}, and their loadings are all subject to estimation errors, TAMLc is very sensitive and its performance can be poor under large estimation errors. As will be seen later in the empirical studies, TAMLc indeed performs very poorly when applied to real market data.

Unlike the conventional wisdom, shrinking towards the current portfolio does not appear to be very effective and is usually dominated by shrinking towards the equal-weight portfolio. The latter approach renders far less volatile portfolios and involves lower transaction costs, leading to robust performance especially under high uncertainty and transaction costs.

The turnover aversion models can be extended by incorporating both w_{ew} and w_{t-}:

$$w^*(\delta, \kappa) = (1-\kappa)\left(\frac{\gamma}{\gamma+\delta} w^* + \frac{\delta}{\gamma+\delta} w_0\right) + \kappa w_{t-}. \tag{34}$$

Figure 4 presents simulation results from this extension. Solid lines represent the case of $\kappa = 0$ and are the same as those in Figure 3. Incorporating both w_{ew} and w_{t-} further improves performance especially when T is small. Nevertheless, the gain from the inclusion of w_{t-} diminishes with δ.



(a) KZ

(b) TZML

(c) TZKZ

(d) TAML

Figure 4: Optimal Turnover Aversion from w_{ew} and w_{t-}: 30 bp Transaction Costs

This figure displays the certainty equivalents of four extended portfolio strategies (Equation (34)) for different values of δ and estimation window sizes in the presence of 30 bp transaction costs. κ is the loading on w_{t-}. Portfolios are assumed to be rebalanced monthly and managed for ten years. The vertical axis represents the normalized certainty equivalent averaged across the datasets; D1, D2, D5, and D8.

4.2 Variance Minimization

Table 7 reports the results from variance minimization. The top panel reports expected variances and the bottom panel reports sample variances from the finite investment horizon.

The standard global minimum-variance portfolio (MV) performs well even for a small T: its expected variance is 27.6% higher than the *ex-post* optimal value when $T = 60$ and only 11.5% higher when $T = 120$. Still, TAMV(0) yields consistently lower variances across all window sizes. TAMV(0) also has smaller standard errors. Meanwhile, incorporating turnover aversion seems to have an adverse effect on variance minimization. This is perhaps because the estimation error of the covariance matrix is not large enough to benefit from turnover aversion. Both TAMVc(0) and TAMVc(1) perform comparably to TAMV(0).

Table 7: Expected and Sample Variances

This table reports the mean and standard error of the variances of selected portfolios, obtained from 10,000 iterations. In the second panel, portfolios are assumed to be rebalanced monthly and managed for ten years. Variances are normalized by that of the *ex-post* global minimum-variance portfolio, MV*. The reported values are the averages across the datasets; D1, D2, D5, and D8. The numbers in the header are estimation window sizes.

	Mean					Standard Error				
	60	120	240	360	600	60	120	240	360	600
MV*	1.000	1.000	1.000	1.000	1.000	0.000	0.000	0.000	0.000	0.000
MV	1.276	1.115	1.054	1.035	1.020	0.127	0.049	0.022	0.014	0.008
TAMV(0)	1.220	1.103	1.051	1.034	1.020	0.099	0.044	0.021	0.014	0.008
TAMV(1)	1.281	1.230	1.202	1.192	1.183	0.071	0.046	0.030	0.023	0.018

(a) Expected Variance

	Mean					Standard Error				
	60	120	240	360	600	60	120	240	360	600
MV*	1.000	1.000	1.000	1.000	1.000	0.129	0.129	0.130	0.130	0.129
MV	1.277	1.116	1.054	1.035	1.021	0.176	0.146	0.137	0.134	0.131
TAMV(0)	1.220	1.103	1.051	1.034	1.020	0.163	0.144	0.137	0.134	0.131
TAMV(1)	1.280	1.230	1.201	1.192	1.183	0.159	0.154	0.153	0.153	0.152
TAMVc(0)	1.240	1.106	1.051	1.033	1.020	0.172	0.145	0.137	0.134	0.131
TAMVc(1)	1.222	1.102	1.051	1.034	1.021	0.169	0.145	0.137	0.134	0.132

(b) Sample Variance

5 Empirical Studies

5.1 Portfolio Construction and Evaluation

Portfolios are rebalanced every month during the sample period based on the mean and covariance estimates obtained from the rolling estimation window of size $T = 60$, 120, or 240. Monthly portfolio returns and performance measures are then calculated.

It is nontrivial to compare different portfolio models on a level playing field. DeMiguel et al. (2009) compare the risky portfolios derived from the optimal portfolios by normalizing the risky asset weights. This method however gives an unfair disadvantage to some models that are not designed to maximize the Sharpe ratio. This issue is discussed in detail in

Kan et al. (2016) for the Kan and Zhou model. Another, more subtle and often overlooked problem is that if the optimal weight of the risky portfolio (sum of the risky asset weights) is negative, the maximum Sharpe ratio portfolio does not exist and the naïve scaling whereby the weights are divided by their sum leads to the minimum Sharpe ratio portfolio as illustrated in Appendix E. In this case, the original portfolio and the derived risky portfolio have opposite exposures to the risky assets and their performances will differ considerably.[13] As shown in Table 2, negative risky portfolio weights are indeed common especially when T is small or in D6 and D7.

Comparing utility maximizing portfolios has its own problem as the results depend on the choice of the risk aversion coefficient. Besides, for those portfolio strategies that do not take the mean into account, *e.g.*, EW, MV, and VT, adjusting the weights so that the utility is maximized conflicts their nature as the adjustment involves the mean.

In this paper, portfolios are constrained so that they have the same *ex-ante* variance (variance targeting). This creates a comparably level playing field for evaluation without favoring a particular model. Imposing a risk constraint is also common in practical asset allocation. If the *ex-ante* variance of an optimal portfolio is $\hat{\sigma}_p^2$ and the target variance is $\sigma_{\max}^2$, the constraint can be satisfied by scaling the portfolio weights with $\sigma_{\max}/\hat{\sigma}_p$.[14] In the empirical studies, the target variance is defined as the variance of the equal-weight portfolio over the entire sample period. Kirby and Ostdiek (2012) adjust portfolios so that they have the same expected return as that of the equal-weight portfolio. While this is similar to variance targeting, the results may be unreliable due to the sizeable estimation error in mean. Imposing constraints on the mean is also less common in practice.

As a robustness test, utility maximizing portfolios are also compared. In this case, the portfolios that are not designed to maximize utility, *i.e.*, EW, MV, VT, TMV, and TAMV, are compared without adjusting the weights so as to maximize utility.[15] Ironically, these models turn out to perform better unadjusted.

Portfolios are evaluated using four performance measures: certainty equivalent (CE),

[13] For this reason, DeMiguel et al. (2009) normalize portfolio weights by the absolute value of their sum. In this case, however, the normalized portfolio still includes the risk-free asset.

[14] If the optimal portfolio has a short position on the risky portfolio, it is scaled by $-\sigma_{\max}/\hat{\sigma}_p$. See Appendix E for details.

[15] One exception is TAMV+. TAMV+ solves the usual utility maximization problem employing the mean implied by the unconstrained TAMV.

Sharpe ratio (SR), turnover (TO), and leverage (LV). These are defined as follows:

$$\text{CE} = \bar{r}_p - \frac{\gamma}{2} s_p^2; \tag{35}$$

$$\text{SR} = \frac{\bar{r}_p}{s_p}; \tag{36}$$

$$\text{TO} = \frac{1}{NT_o} \sum_{t=1}^{T_o} \sum_{i=1}^{N} |w_{i,t} - w_{i,t-}|; \tag{37}$$

$$\text{LV} = \frac{1}{T_o} \sum_{t=1}^{T_o} \sum_{i=1}^{N} |w_{i,t}|, \tag{38}$$

where $\bar{r}_p$ and s_p are the mean and standard deviation of the portfolio returns over the sample period, T_o is the sample size, and $w_{i,t-}$ and $w_{i,t}$ are the weights of asset i immediately before and after rebalancing at time t. For certainty equivalent, $\gamma = 3$ is used. Note that turnover is not penalized when portfolio performance is evaluated. It is assumed that, while the investor's aversion to turnover plays a role during portfolio rebalancing, the utility of the investor depends entirely on the mean and variance of the portfolio returns after rebalancing. To assess the effect of transaction costs, SR and CE are calculated both before and after transaction costs assuming transaction costs of 10 bp for buying and selling risky assets and 0 bp for the risk-free asset. LV is calculated to gauge the feasibility of the portfolios. Highly leveraged portfolios are not desirable and unrealistic to many investors. While imposing constraints on the asset weights can yield more feasible portfolios, its impact on portfolio performance will be nontrivial, making the performance attributable to the unique characteristics of individual models indistinguishable. Therefore, portfolios are evaluated without weight constraints and LV is calculated to supplement the results. In addition, the results under the weight constraint, $|w_i| \leq 0.5$, are provided in Internet Appendix for comparison.

Given thirteen datasets and four performance measures, a coherent evaluation and ranking of the models is a formidable task. To facilitate evaluation, SR and CE are normalized by the SR and CE of the *ex-post* optimal portfolio. Normalization helps measure the utility loss and allows us to compare the performance of a portfolio across datasets. The normalized measures are averaged across the datasets in order to generate a single measure for each performance metric. Despite the fact that the summary statistics across datasets depend on the choice of datasets, this provides a convenient way of comparing models. The empirical analysis in this section is primarily based on the summary statistics and the results from each dataset are referred to when necessary.

Forty-seven variants of the portfolio models in Table 3 are tested on the thirteen datasets in Table 1, and the summary results on CE, TO, and LV are reported in Table 8 (variance targeting) and Table 9 (utility maximization) for $T = 120$. In each table, the columns 'Mean' and 'Std' are the mean and standard deviation of the normalized measures across the datasets, and the column 'P(>EW)', referred to as outperformance ratio, is the proportion of the datasets in which a model outperforms EW. The outperformance ratio is calculated for each of the mean CE and SR before/after transaction costs. The numbers next to each column are the ranks of the models based on the corresponding performance measure.

Internet Appendix provides full results including; results on SR, results from variance targeting with weight constraints, results from $T = 60$ and 240, results from datasets without factor portfolios, sub-period analysis results, and dataset-level results.

5.2 Main Findings

The most remarkable finding from the empirical studies is the performance enhancement from turnover aversion. Consistent with the simulation results, incorporating turnover aversion improves portfolio performance significantly, not only for the proposed models, TAML and TAMV, but also for the existing models; KZ, TZML, and TZKZ. Recall that these models already address parameter uncertainty before incorporating turnover aversion. In general, a model involving three portfolios (KZ and TZKZ) outperforms a model involving two portfolios (TZML and TAML), and KZ is revealed to perform best overall when augmented with turnover aversion.

Another important finding is the sharp contrast between the turnover aversion models with $w_0 = w_{ew}$ and those with $w_0 = w_{t-}$. As anticipated from the simulation studies, incorporating w_{t-} is not as effective as incorporating w_{ew}, both before and after transaction costs.[16] When shrunk towards w_{t-}, KZ, TZML, and TZKZ perform slightly worse than their base models before transaction costs and slightly better after transaction costs. On the other hand, shrinking towards w_{ew} improves the performance substantially both before and after transaction costs. The results from TAML is particularly noteworthy. Contrary to the simulation results where TAML with $w_0 = w_{t-}$ performs better, it performs very poorly when applied to the real market data significantly underperforming TAML with $w_0 = w_{ew}$. The contrast becomes more evident in utility maximization (Table 9): incorporating w_{ew} considerably improves the performance of all models, whereas incorporating w_{t-} is almost harmful. When a portfolio is subject to large estimation errors, the current portfolio can be substantially different from the optimal portfolio and penalizing turnover from it does not necessarily improve performance even after transaction costs.

As shown in Table 9, the performance of most portfolios deteriorates without the variance

[16] As the models with $w_0 = w_{t-}$ perform poorly, only one case ($\delta = 1$) is reported.

Table 8: Certainty Equivalent: Variance Targeting, T = 120

This table reports the normalized CE's from variance targeting. 'Mean' and 'Std' are the mean and standard deviation of the normalized CE's across the datasets in Table 1, and 'P(>EW)' (outperformance ratio) is the proportion of the datasets in which a portfolio outperforms EW. 'TO' and 'LV' are respectively the turnover and leverage defined in Section 5.1, averaged across the datasets. The numbers next to each column are the ranks of the portfolios based on the measure of that column. Details of the portfolio models can be found in Section 3.2.

	Before Transaction Cost						After Transaction Cost						TO		LV	
	Mean		Std		P(>EW)		Mean		Std		P(>EW)					
W*	1.000		0.000		1.000		1.000		0.000		1.000		0.019		8.399	
EW	0.242	46	0.210	10	0.000	46	0.243	41	0.211	9	0.000	46	0.003	1	1.045	2
ML	0.649	24	0.396	44	0.769	33	0.489	24	0.455	40	0.692	31	0.116	46	17.430	45
ML+	0.475	27	0.196	8	1.000	1	0.470	25	0.198	8	1.000	1	0.011	13	1.249	10
MV	0.315	41	0.348	38	0.692	35	0.142	45	0.495	42	0.462	45	0.114	44	17.748	46
MV+	0.324	39	0.227	12	0.692	35	0.320	38	0.231	12	0.615	36	0.009	9	1.632	14
VT	0.252	45	0.194	7	0.769	33	0.253	40	0.196	7	0.692	31	0.004	3	1.153	3
OC	0.717	22	0.304	25	0.846	31	0.588	22	0.354	32	0.692	31	0.096	40	14.261	36
OC+	0.398	34	0.158	1	0.923	20	0.392	35	0.158	1	0.923	10	0.012	14	1.000	1
TML(.05)	0.705	23	0.383	42	0.846	31	0.601	21	0.438	38	0.692	31	0.079	26	10.608	19
TML+(.05)	0.466	28	0.180	4	1.000	1	0.466	26	0.181	4	1.000	1	0.009	8	1.166	5
TMLc(.05)	0.533	25	0.456	45	0.615	40	0.459	27	0.499	43	0.615	36	0.057	17	15.439	39
TMLc+(.05)	0.419	33	0.186	6	0.923	20	0.425	33	0.188	6	0.923	10	0.003	2	1.230	9
TMV(.05)	0.332	38	0.394	43	0.615	40	0.196	43	0.609	46	0.615	36	0.088	35	13.356	32
TMV+(.05)	0.336	37	0.259	15	0.692	35	0.336	37	0.264	14	0.692	31	0.006	6	1.555	12
TMVc(.05)	0.304	43	0.356	40	0.692	35	0.209	42	0.480	41	0.615	36	0.066	19	15.258	38
TMVc+(.05)	0.316	40	0.268	16	0.538	45	0.317	39	0.274	16	0.538	42	0.005	5	1.574	13
KZ	0.742	14	0.329	35	1.000	1	0.586	23	0.381	36	0.846	22	0.114	45	17.334	44
KZ(1)	0.792	3	0.304	26	1.000	1	0.656	8	0.344	29	0.846	22	0.103	42	15.855	40
KZ(2)	0.808	1	0.285	21	1.000	1	0.690	3	0.316	22	1.000	1	0.092	36	14.316	37
KZ(3)	0.804	2	0.271	18	1.000	1	0.701	1	0.295	18	1.000	1	0.082	29	12.970	28
KZc(1)	0.736	17	0.327	34	1.000	1	0.602	20	0.371	33	0.846	22	0.101	41	17.332	43
TZML	0.735	18	0.348	37	0.923	20	0.613	19	0.384	37	0.769	28	0.094	38	13.244	31
TZML(1)	0.743	13	0.323	31	0.923	20	0.637	16	0.351	31	0.846	22	0.084	33	11.969	25
TZML(2)	0.740	16	0.303	24	1.000	1	0.647	13	0.326	24	0.923	10	0.075	24	10.891	22
TZML(3)	0.730	20	0.287	22	1.000	1	0.648	12	0.306	20	0.923	10	0.068	21	9.976	18
TZMLc(1)	0.730	19	0.348	39	0.923	20	0.625	17	0.379	35	0.769	28	0.083	32	13.241	30
TZKZ	0.784	5	0.308	27	1.000	1	0.660	7	0.338	27	0.846	22	0.096	39	14.148	35
TZKZ(1)	0.791	4	0.285	20	1.000	1	0.689	4	0.306	21	0.923	10	0.082	28	12.284	27
TZKZ(2)	0.776	7	0.269	17	1.000	1	0.690	2	0.284	17	1.000	1	0.071	22	10.830	21
TZKZ(3)	0.755	8	0.256	14	1.000	1	0.682	5	0.267	15	1.000	1	0.063	18	9.680	16
TZKZc(1)	0.777	6	0.310	28	0.923	20	0.673	6	0.334	26	0.923	10	0.083	30	14.141	34
TAML(0)	0.743	12	0.339	36	0.923	20	0.623	18	0.372	34	0.769	28	0.093	37	13.103	29
TAML(1)	0.747	10	0.317	30	0.923	20	0.643	15	0.343	28	0.846	22	0.083	31	11.811	24
TAML(2)	0.741	15	0.299	23	1.000	1	0.649	9	0.320	23	0.923	10	0.075	23	10.728	20
TAML(3)	0.730	21	0.283	19	1.000	1	0.649	11	0.300	19	0.923	10	0.068	20	9.812	17
TAMLc(1)	0.504	26	0.520	46	0.615	40	0.441	32	0.580	45	0.615	36	0.047	16	14.131	33
TAMLK(0)	0.754	9	0.327	33	0.923	20	0.645	14	0.350	30	0.923	10	0.087	34	12.157	26
TAMLK(1)	0.744	11	0.311	29	1.000	1	0.649	10	0.328	25	0.923	10	0.078	25	10.933	23
TAML+(0)	0.462	29	0.179	2	1.000	1	0.459	28	0.180	2	1.000	1	0.011	12	1.202	7
TAML+(1)	0.458	30	0.180	3	1.000	1	0.456	29	0.180	3	1.000	1	0.010	11	1.186	6
TAML+(3)	0.447	31	0.182	5	1.000	1	0.446	30	0.183	5	1.000	1	0.009	10	1.162	4
TAMV(0)	0.292	44	0.359	41	0.615	40	0.121	46	0.520	44	0.538	42	0.112	43	17.067	41
TAMV(1)	0.360	36	0.235	13	0.615	40	0.339	36	0.237	13	0.615	36	0.021	15	3.452	15
TAMVc(1)	0.310	42	0.325	32	0.692	35	0.195	44	0.451	39	0.538	42	0.079	27	17.127	42
TAMV+(0)	0.439	32	0.210	9	0.923	20	0.442	31	0.213	10	0.923	10	0.006	7	1.539	11
TAMV+(1)	0.395	35	0.213	11	0.923	20	0.399	34	0.214	11	0.923	10	0.005	4	1.217	8

Table 9: Certainty Equivalent: Utility Maximization, T = 120

This table reports the normalized CE's from utility maximization. 'Mean' and 'Std' are the mean and standard deviation of the normalized CE's across the datasets in Table 1, and 'P(>EW)' (outperformance ratio) is the proportion of the datasets in which a portfolio outperforms EW. 'TO' and 'LV' are respectively the turnover and leverage defined in Section 5.1, averaged across the datasets. The numbers next to each column are the ranks of the portfolios based on the measure of that column. Details of the portfolio models can be found in Section 3.2.

	Before Transaction Cost						After Transaction Cost						TO		LV	
	Mean		Std		P(>EW)		Mean		Std		P(>EW)					
W*	1.000		0.000		1.000		1.000		0.000		1.000		0.095		20.840	
EW	0.240	37	0.136	3	0.000	42	0.257	31	0.141	3	0.000	41	0.002	2	1.000	3
ML	-2.919	45	2.794	45	0.000	42	-4.599	46	4.384	45	0.000	41	1.162	46	82.766	46
ML+	0.038	40	0.256	39	0.385	38	0.002	40	0.277	34	0.231	38	0.045	20	3.054	16
MV	0.270	29	0.169	14	0.769	22	0.259	29	0.163	11	0.615	27	0.027	18	5.069	20
MV+	0.251	32	0.144	7	0.692	29	0.265	25	0.148	8	0.692	18	0.004	7	1.000	3
VT	0.248	34	0.129	2	0.923	1	0.266	24	0.134	2	0.923	1	0.002	3	1.000	3
OC	-2.576	44	2.691	44	0.000	42	-4.025	44	4.209	44	0.000	41	0.974	45	71.655	44
OC+	0.251	33	0.123	1	0.615	35	0.258	30	0.125	1	0.615	27	0.012	9	1.000	3
TML(.05)	-1.532	43	2.230	42	0.154	39	-2.520	43	3.297	42	0.000	41	0.703	43	54.997	43
TML+(.05)	0.217	38	0.201	20	0.615	35	0.209	33	0.207	19	0.462	35	0.029	19	2.365	12
TMLc(.05)	-2.981	46	3.068	46	0.000	42	-4.379	45	4.665	46	0.000	41	0.887	44	75.925	45
TMLc+(.05)	-0.006	41	0.222	30	0.154	39	-0.027	41	0.242	26	0.154	39	0.024	16	2.907	14
TMV(.05)	0.281	25	0.171	16	0.769	22	0.287	19	0.175	17	0.769	6	0.012	12	3.018	15
TMV+(.05)	0.254	31	0.138	5	0.769	22	0.271	23	0.143	5	0.769	6	0.002	4	1.000	7
TMVc(.05)	0.263	30	0.160	11	0.769	22	0.273	22	0.166	14	0.692	18	0.009	8	4.055	17
TMVc+(.05)	0.246	36	0.146	8	0.615	35	0.263	26	0.152	9	0.615	27	0.001	1	1.000	8
KZ	0.349	20	0.266	41	0.692	29	0.034	39	0.395	41	0.385	36	0.344	41	33.519	41
KZ(1)	0.564	11	0.250	38	0.846	17	0.385	13	0.314	37	0.692	18	0.226	35	25.197	35
KZ(2)	0.615	2	0.231	36	0.923	1	0.501	7	0.270	31	0.769	6	0.166	29	20.208	30
KZ(3)	0.617	1	0.214	25	0.923	1	0.537	1	0.239	25	0.769	6	0.131	25	16.885	26
KZc(1)	0.331	24	0.264	40	0.692	29	0.050	38	0.382	40	0.385	36	0.310	40	33.459	40
TZML	0.419	18	0.228	34	0.692	29	0.155	37	0.335	39	0.538	32	0.302	39	28.221	39
TZML(1)	0.557	13	0.234	37	0.846	17	0.406	12	0.291	35	0.692	18	0.199	33	21.240	32
TZML(2)	0.581	5	0.221	28	0.923	1	0.483	9	0.259	29	0.769	6	0.147	27	17.055	27
TZML(3)	0.573	7	0.205	23	0.923	1	0.505	5	0.232	24	0.769	6	0.116	24	14.268	24
TZMLc(1)	0.401	19	0.228	33	0.692	29	0.169	36	0.326	38	0.615	27	0.270	36	28.172	38
TZKZ	0.545	14	0.224	32	0.923	1	0.373	15	0.270	32	0.692	18	0.224	34	23.142	34
TZKZ(1)	0.602	3	0.219	27	0.923	1	0.505	4	0.242	27	0.769	6	0.151	28	17.428	28
TZKZ(2)	0.594	4	0.204	22	0.923	1	0.532	2	0.218	20	0.769	6	0.113	23	14.004	23
TZKZ(3)	0.571	9	0.190	18	0.923	1	0.530	3	0.200	18	0.846	5	0.090	21	11.725	21
TZKZc(1)	0.530	15	0.222	29	0.846	17	0.384	14	0.262	30	0.692	18	0.198	32	23.103	33
TAML(0)	0.442	17	0.224	31	0.846	17	0.193	35	0.307	36	0.538	32	0.290	38	27.359	37
TAML(1)	0.562	12	0.229	35	0.846	17	0.419	11	0.273	33	0.692	18	0.193	31	20.595	31
TAML(2)	0.580	6	0.217	26	0.923	1	0.486	8	0.248	28	0.769	6	0.143	26	16.541	25
TAML(3)	0.569	10	0.202	21	0.923	1	0.504	6	0.224	21	0.769	6	0.113	22	13.841	22
TAMLc(1)	-1.346	42	2.408	43	0.000	42	-1.972	42	3.345	43	0.000	41	0.387	42	42.799	42
TAMLK(0)	0.470	16	0.201	19	0.923	1	0.251	32	0.224	22	0.538	32	0.270	37	25.418	36
TAMLK(1)	0.573	8	0.212	24	0.923	1	0.450	10	0.226	23	0.769	6	0.180	30	19.145	29
TAML+(0)	0.344	22	0.171	17	0.923	1	0.356	17	0.169	15	0.923	1	0.020	15	1.594	11
TAML+(1)	0.346	21	0.165	12	0.923	1	0.359	16	0.164	12	0.923	1	0.016	14	1.440	10
TAML+(3)	0.334	23	0.155	10	0.923	1	0.349	18	0.155	10	0.923	1	0.013	13	1.289	9
TAMV(0)	0.271	28	0.170	15	0.769	22	0.263	27	0.166	13	0.615	27	0.025	17	4.689	18
TAMV(1)	0.279	26	0.148	9	0.769	22	0.287	20	0.147	7	0.769	6	0.012	10	2.434	13
TAMVc(1)	0.273	27	0.168	13	0.769	22	0.279	21	0.171	16	0.692	18	0.012	11	4.905	19
TAMV+(0)	0.192	39	0.138	4	0.154	39	0.202	34	0.143	4	0.154	39	0.003	5	0.594	1
TAMV+(1)	0.247	35	0.140	6	0.692	29	0.261	28	0.143	6	0.692	18	0.004	6	0.862	2

constraint and outperforming EW becomes far more challenging. This is because utility maximizing portfolios are more sensitive to the mean estimate. In fact, EW also performs considerably worse if its weights are adjusted so as to maximize utility (unreported). The performance of the optimal portfolios that ignore parameter uncertainty, *e.g.*, ML and OC, drops most. KZ, TZML, TZKZ, and TAML without turnover aversion perform relatively better, but they also experience a nontrivial performance drop. On the other hand, when augmented with turnover aversion, these models perform robustly and their normalized CE's are not particularly lower than those from variance targeting. It is also worth noting that a higher degree of turnover aversion is required to maximize performance in utility maximization.

Detailed analyses of the results are given in Appendix F and the remainder of this section is devoted to the calibration of δ.

5.3 Calibration of δ

So far, the turnover aversion models have been assessed using different values of δ. However, δ needs to be determined beforehand in order to implement the models. This section proposes a simple, but effective calibration method. The procedure is as follows:

1. For the first ten months into the sample period, δ is set to 3, 2, or 1 respectively when $T = 60$, 120, or 240.

2. When $t > 10$, δ is calibrated each month so that the CE during $1, \ldots, t-1$ is maximized. The optimal δ is found via line search spanning the range $[0, 10]$.[17]

3. The above step is repeated for the CE after transaction costs.

The turnover aversion coefficient δ is calibrated for the four models; KZ, TZML, TZKZ, and TAML, and the results are reported in Table 10 (variance targeting) and Table 11 (utility maximization). $\bar{\delta}_b^*$ ($\bar{\delta}_a^*$) denotes the mean of the calibrated δ without (with) transaction costs, and its values are reported under 'Before (After) Cost'. The last two rows of each model are the results from the extension in (34) with $\kappa = 0.1$ and 0.2, respectively. Previous results with a constant δ are also reproduced for comparison. The maximum CE within each model and window size, and P(> EW) = 1 are highlighted in boldface.

The performance enhancement from the calibration is remarkable. The models with a calibrated δ have higher CE's than any of the constant-δ models and outperform EW more

[17]A maximum CE sometimes occurs at $\delta > 10$, but the increment of CE is marginal when δ is large and allowing a larger δ has little effect on the results.

Table 10: Effects of δ Calibration: Variance Targeting

This table compares the turnover aversion models with calibrated δ against those with constant δ. The reported values are the normalized CE's averaged across the datasets in Table 1. $\bar{\delta_b^*}$ ($\bar{\delta_a^*}$) denotes the mean of the calibrated δ without (with) transaction costs, and its values are presented in the columns under 'Before (After) Cost'. The last two rows of each model are the results from the extension in (34) with $\kappa = 0.1$ and 0.2. The boldface figures under 'Mean CE' refer to the maximum CE within each model and window size, and those under 'P(>EW)' refer to the cases in which EW is outperformed in all datasets.

	Mean CE						P(>EW)					
	Before Cost			After Cost			Before Cost			After Cost		
	60	120	240	60	120	240	60	120	240	60	120	240
KZ	0.402	0.742	0.668	-0.041	0.586	0.597	0.69	**1.00**	0.85	0.46	0.85	0.69
KZ(1)	0.545	0.792	0.692	0.149	0.656	0.633	0.92	**1.00**	0.92	0.54	0.85	0.85
KZ(2)	0.623	0.808	0.692	0.276	0.690	0.642	0.92	**1.00**	**1.00**	0.54	**1.00**	0.92
KZ(3)	0.655	0.804	0.682	0.351	0.701	0.640	0.92	**1.00**	**1.00**	0.69	**1.00**	0.92
$\bar{\delta_b^*}$, $\bar{\delta_a^*}$	4.5	2.5	1.8	6.4	3.7	2.7						
KZ(δ_b^*)	**0.774**	**0.873**	0.733	0.524	0.764	0.676	**1.00**	**1.00**	**1.00**	0.85	**1.00**	**1.00**
KZ(δ_a^*)	0.763	0.857	**0.734**	**0.559**	**0.765**	**0.684**	**1.00**	**1.00**	**1.00**	0.92	**1.00**	**1.00**
KZ(δ_a^*,.1)	0.742	0.845	0.728	0.554	0.760	0.680	**1.00**	**1.00**	**1.00**	0.92	**1.00**	**1.00**
KZ(δ_a^*,.2)	0.729	0.833	0.720	0.556	0.754	0.675	**1.00**	**1.00**	**1.00**	0.92	**1.00**	**1.00**
TZML	0.653	0.735	0.653	0.350	0.613	0.594	0.85	0.92	0.77	0.69	0.77	0.69
TZML(1)	0.671	0.743	0.653	0.405	0.637	0.603	0.92	0.92	0.77	0.77	0.85	0.69
TZML(2)	0.666	0.740	0.646	0.431	0.647	0.604	**1.00**	**1.00**	0.92	0.77	0.92	0.77
TZML(3)	0.659	0.730	0.635	0.449	0.648	0.598	**1.00**	**1.00**	0.92	0.77	0.92	0.77
$\bar{\delta_b^*}$, $\bar{\delta_a^*}$	3.2	2.3	1.8	5.5	3.9	3.1						
TZML(δ_b^*)	**0.771**	**0.797**	0.691	0.545	0.703	0.636	**1.00**	**1.00**	**1.00**	0.85	**1.00**	0.77
TZML(δ_a^*)	0.743	0.793	**0.693**	**0.573**	**0.715**	**0.648**	**1.00**	**1.00**	0.92	0.85	**1.00**	0.92
TZML(δ_a^*,.1)	0.711	0.784	0.688	0.552	0.710	0.645	**1.00**	**1.00**	0.92	0.85	**1.00**	0.92
TZML(δ_a^*,.2)	0.703	0.777	0.684	0.556	0.708	0.643	**1.00**	**1.00**	0.92	0.85	**1.00**	0.92
TZKZ	0.597	0.784	0.685	0.241	0.660	0.631	0.92	**1.00**	**1.00**	0.54	0.85	0.85
TZKZ(1)	0.652	0.791	0.674	0.357	0.689	0.632	0.92	**1.00**	**1.00**	0.69	0.92	0.85
TZKZ(2)	0.662	0.776	0.657	0.413	0.690	0.622	0.92	**1.00**	**1.00**	0.77	**1.00**	0.92
TZKZ(3)	0.659	0.755	0.638	0.446	0.682	0.609	**1.00**	**1.00**	**1.00**	0.85	**1.00**	0.92
$\bar{\delta_b^*}$, $\bar{\delta_a^*}$	2.9	1.5	0.9	5.0	2.7	1.6						
TZKZ(δ_b^*)	**0.770**	**0.829**	0.704	0.530	0.730	0.653	**1.00**	**1.00**	**1.00**	0.85	**1.00**	**1.00**
TZKZ(δ_a^*)	0.745	0.821	**0.708**	**0.576**	**0.739**	**0.660**	**1.00**	**1.00**	**1.00**	0.92	**1.00**	**1.00**
TZKZ(δ_a^*,.1)	0.720	0.810	0.703	0.563	0.734	0.658	**1.00**	**1.00**	**1.00**	0.92	**1.00**	**1.00**
TZKZ(δ_a^*,.2)	0.708	0.802	0.698	0.563	0.731	0.655	**1.00**	**1.00**	**1.00**	0.92	**1.00**	**1.00**
TAML(0)	0.639	0.743	0.649	0.337	0.623	0.589	0.85	0.92	0.69	0.77	0.77	0.69
TAML(1)	0.639	0.747	0.649	0.373	0.643	0.598	0.85	0.92	0.69	0.77	0.85	0.69
TAML(2)	0.635	0.741	0.641	0.400	0.649	0.598	**1.00**	**1.00**	0.85	0.77	0.92	0.77
TAML(3)	0.621	0.730	0.630	0.411	0.649	0.593	**1.00**	**1.00**	0.85	0.77	0.92	0.77
$\bar{\delta_b^*}$, $\bar{\delta_a^*}$	3.2	2.1	1.9	5.6	3.7	3.3						
TAML(δ_b^*)	**0.739**	**0.800**	0.688	0.518	0.704	0.633	**1.00**	**1.00**	**1.00**	0.85	**1.00**	0.77
TAML(δ_a^*)	0.704	0.800	**0.690**	**0.537**	**0.720**	**0.644**	**1.00**	**1.00**	**1.00**	0.85	**1.00**	0.85
TAML(δ_a^*,.1)	0.685	0.792	0.685	0.530	0.717	0.641	**1.00**	**1.00**	**1.00**	0.85	**1.00**	0.85
TAML(δ_a^*,.2)	0.674	0.786	0.680	0.529	0.715	0.638	**1.00**	**1.00**	**1.00**	0.85	**1.00**	0.85

frequently. For instance, when $T = 120$, the mean CE of KZ is 0.742 before transaction costs and it increases to 0.808 when $\delta = 2$, whereas that of KZ(δ_b^*) is 0.873. The corresponding

Table 11: Effects of δ Calibration: Utility Maximization

This table compares the turnover aversion models with calibrated δ against those with constant δ. The reported values are the normalized CE's averaged across the datasets in Table 1. $\bar{\delta}_b^*$ ($\bar{\delta}_a^*$) denotes the mean of the calibrated δ without (with) transaction costs, and its values are presented in the columns under 'Before (After) Cost'. The last two rows of each model are the results from the extension in (34) with $\kappa = 0.1$ and 0.2. The boldface figures under 'Mean CE' refer to the maximum CE within each model and window size, and those under 'P(>EW)' refer to the cases in which EW is outperformed in all datasets.

	Mean CE						P(>EW)					
	Before Cost			After Cost			Before Cost			After Cost		
	60	120	240	60	120	240	60	120	240	60	120	240
KZ	-0.310	0.349	0.234	-0.994	0.034	0.059	0.08	0.69	0.54	0.00	0.38	0.38
KZ(1)	0.157	0.564	0.424	-0.285	0.385	0.337	0.54	0.85	0.69	0.15	0.69	0.54
KZ(2)	0.331	0.615	0.478	0.015	0.501	0.430	0.69	0.92	0.77	0.38	0.77	0.62
KZ(3)	0.404	0.617	0.488	0.161	0.537	0.462	0.69	0.92	0.77	0.46	0.77	0.69
$\bar{\delta}_b^*$, $\bar{\delta}_a^*$	4.5	2.7	3.2	6.5	4.3	4.3						
KZ(δ_b^*)	0.506	0.669	0.559	0.298	0.542	0.496	**1.00**	**1.00**	**1.00**	0.54	0.85	0.77
KZ(δ_a^*)	**0.512**	**0.676**	**0.578**	0.375	**0.600**	**0.539**	**1.00**	**1.00**	**1.00**	0.69	**1.00**	**1.00**
KZ(δ_a^*,.1)	0.501	0.665	0.566	**0.376**	0.594	0.529	**1.00**	**1.00**	**1.00**	0.69	**1.00**	0.85
KZ(δ_a^*,.2)	0.489	0.652	0.553	0.375	0.587	0.518	**1.00**	**1.00**	**1.00**	0.69	**1.00**	0.85
TZML	0.022	0.419	0.317	-0.514	0.155	0.173	0.31	0.69	0.54	0.08	0.54	0.46
TZML(1)	0.302	0.557	0.453	-0.048	0.406	0.381	0.69	0.85	0.69	0.23	0.69	0.62
TZML(2)	0.398	0.581	0.485	0.146	0.483	0.446	0.69	0.92	0.77	0.46	0.77	0.62
TZML(3)	0.432	0.573	0.485	0.238	0.505	0.465	0.77	0.92	0.77	0.62	0.77	0.77
$\bar{\delta}_b^*$, $\bar{\delta}_a^*$	3.7	2.1	2.8	6.1	3.8	4.1						
TZML(δ_b^*)	0.506	0.627	0.548	0.304	0.492	0.482	0.92	0.92	0.92	0.69	0.69	0.69
TZML(δ_a^*)	**0.510**	**0.639**	**0.564**	0.389	**0.565**	**0.525**	**1.00**	0.92	0.92	0.77	0.85	0.77
TZML(δ_a^*,.1)	0.499	0.628	0.553	**0.389**	0.560	0.517	**1.00**	0.92	0.92	0.77	0.85	0.77
TZML(δ_a^*,.2)	0.487	0.617	0.542	0.388	0.554	0.508	**1.00**	0.92	0.92	0.77	0.85	0.77
TZKZ	0.166	0.545	0.391	-0.233	0.373	0.298	0.54	0.92	0.62	0.08	0.69	0.54
TZKZ(1)	0.356	0.602	0.469	0.095	0.505	0.426	0.69	0.92	0.77	0.38	0.77	0.62
TZKZ(2)	0.415	0.594	0.479	0.226	0.532	0.459	0.77	0.92	0.77	0.62	0.77	0.69
TZKZ(3)	0.432	0.571	0.470	0.284	0.530	0.463	0.77	0.92	0.92	0.62	0.85	0.77
$\bar{\delta}_b^*$, $\bar{\delta}_a^*$	3.0	1.6	2.3	5.3	3.1	3.4						
TZKZ(δ_b^*)	**0.503**	0.642	0.532	0.318	0.527	0.486	**1.00**	**1.00**	**1.00**	0.62	0.85	0.85
TZKZ(δ_a^*)	0.502	**0.646**	**0.549**	**0.394**	**0.580**	**0.520**	**1.00**	**1.00**	**1.00**	0.77	**1.00**	0.92
TZKZ(δ_a^*,.1)	0.491	0.636	0.539	0.392	0.575	0.511	**1.00**	**1.00**	**1.00**	0.77	**1.00**	0.85
TZKZ(δ_a^*,.2)	0.479	0.626	0.528	0.389	0.570	0.502	**1.00**	**1.00**	**1.00**	0.77	**1.00**	0.85
TAML(0)	0.060	0.442	0.330	-0.452	0.193	0.191	0.38	0.85	0.54	0.08	0.54	0.46
TAML(1)	0.307	0.562	0.456	-0.029	0.419	0.386	0.69	0.85	0.77	0.23	0.69	0.54
TAML(2)	0.391	0.580	0.484	0.149	0.486	0.447	0.69	0.92	0.77	0.46	0.77	0.62
TAML(3)	0.420	0.569	0.483	0.233	0.504	0.463	0.77	0.92	0.77	0.62	0.77	0.77
$\bar{\delta}_b^*$, $\bar{\delta}_a^*$	3.8	1.8	2.7	6.2	3.6	4.0						
TAML(δ_b^*)	**0.487**	0.618	0.545	0.293	0.480	0.480	0.92	0.92	0.92	0.69	0.69	0.69
TAML(δ_a^*)	0.487	**0.630**	**0.559**	0.371	**0.556**	**0.523**	**1.00**	0.92	**1.00**	0.77	0.92	0.77
TAML(δ_a^*,.1)	0.478	0.621	0.549	**0.374**	0.553	0.515	**1.00**	0.92	0.92	0.77	0.85	0.77
TAML(δ_a^*,.2)	0.469	0.610	0.539	0.374	0.548	0.507	**1.00**	0.92	0.92	0.77	0.85	0.77

values after transaction costs are respectively 0.586, 0.701 (when $\delta = 3$), and 0.765 (when $\delta = \delta_a^*$). As intended, the maximum CE's before transaction costs are usually associated with δ_b^*, and those after transaction costs are associated with δ_a^*. Meanwhile, augmenting the models with the current portfolio (the last two rows of each model) appears to do more harm than good. This again raises a doubt on the effectiveness of shrinking towards the current portfolio.

The effect of calibration is particularly pronounced in utility maximization: both the CE and outperformance ratio are considerably larger after calibration. For instance, there are several cases where KZ with a calibrated δ outperforms EW in all thirteen datasets after transaction costs ($\delta = \delta_a^*$; $T = 120$ or 240), whereas KZ with a constant δ outperforms EW in ten datasets at most ($\delta = 2$ or 3; $T = 120$).

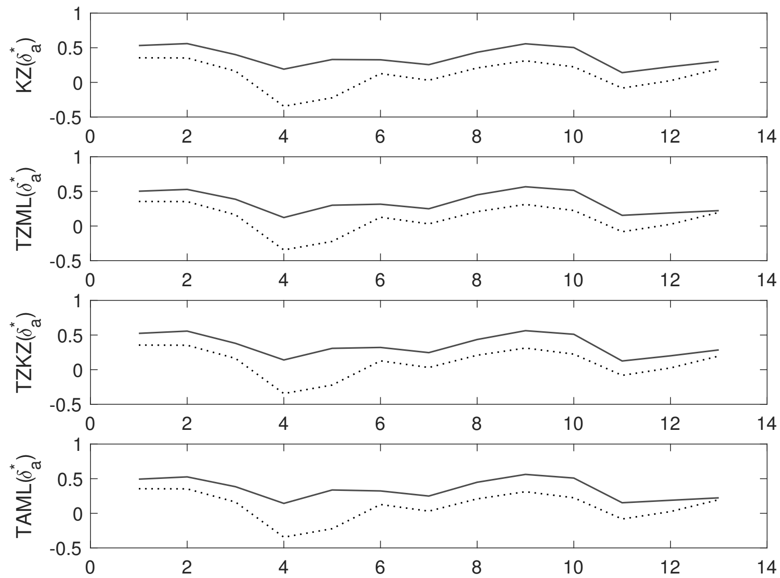
The sub-period analysis results reported in Figure 5 and 6 show that the calibrated models perform consistently over time. In particular, KZ and TZKZ maintain their superior performance across the sub-periods with small variation.

5.4 Robustness Check

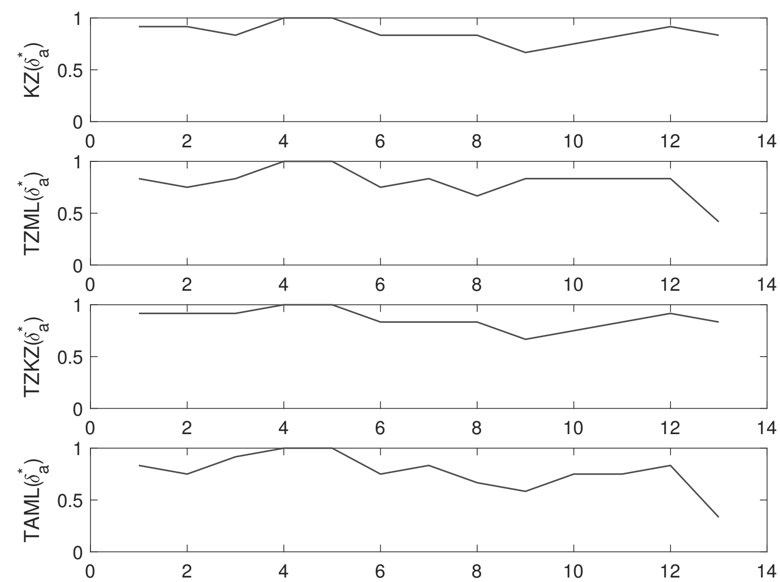
Comprehensive robustness tests are conducted and the results are provided in Internet Appendix. This includes sub-period analyses and tests on ten additional datasets which do not contain the market and factor portfolios. Also included is an additional optimization criterion: variance targeting with weight constraints, $|w_i| \leq 0.5$, on the risky assets. The purpose of this criterion is to generate more realistic portfolios with low leverage.

When the models are tested on the new datasets, the results are qualitatively similar to those presented here, but the resulting portfolios are usually less leveraged. This is because it is no longer possible to short the market and buy other assets. Imposing the weight constraints does not alter the results considerably and the rankings of the models are largely preserved. The CE and SR tend to be slightly smaller with the constraints, but the optimal portfolios perform more consistently across the datasets and outperform EW more frequently. This result is similar to that from the short-sale constrained models, but relaxing the lower bound seems to strike a better balance between robustness and performance. By and large, the conclusions drawn in this paper remain valid in the additional datasets and optimization criterion.

The sub-period analysis shows that the portfolios perform rather consistently across sub-periods in variance targeting. The rankings of the portfolios are largely unchanged and the outperformance ratios are also stable. On the other hand, many portfolios perform inconsistently in utility maximization. For example, ML+ outperforms EW in eleven datasets in the first sub-period but fails to outperform EW in any dataset in the fourth sub-period. In contrast, the turnover aversion models maintain stable, superior performance across sub-periods



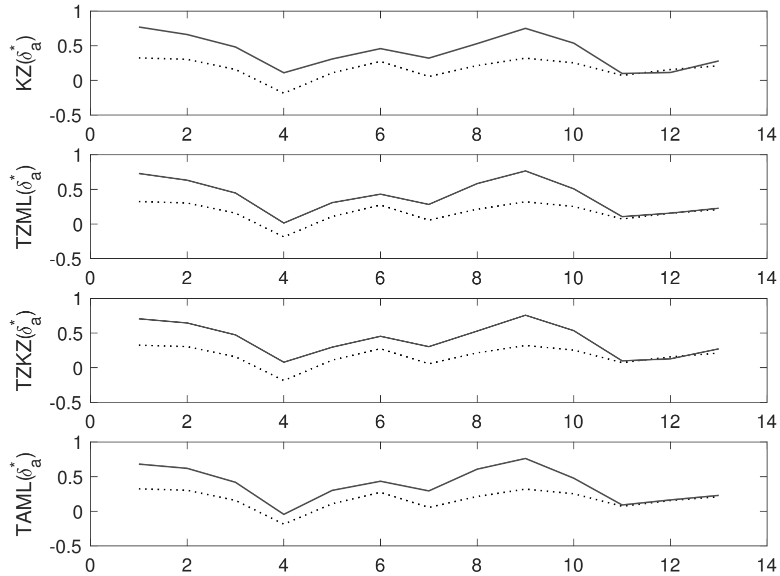
(a) Certainty Equivalent



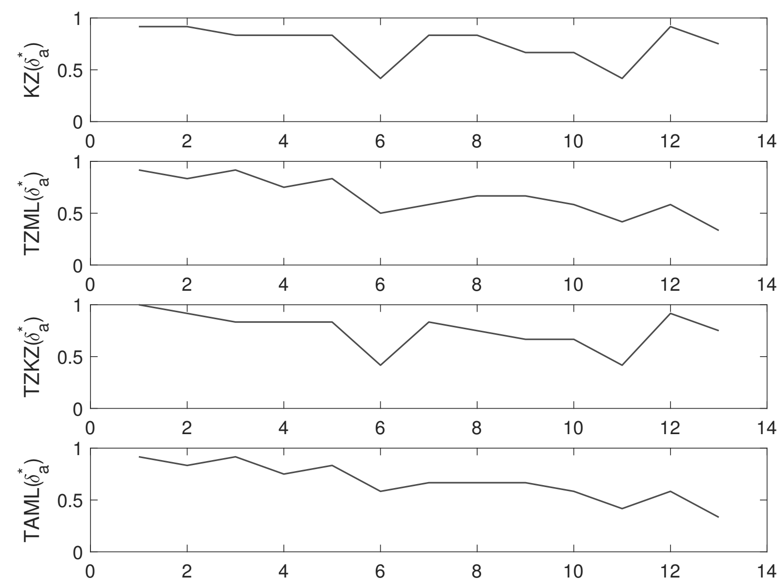
(b) Outperformance, P(>EW)

Figure 5: Sub-Period Performance with Calibrated δ: Variance Targeting

This figure visualizes the performance of the calibrated turnover aversion models in sub-periods (horizontal axis). Each sub-period is ten-year long and five-years apart from each other, except the last sub-period, SP13, which ends in 2015.12. The upper chart displays the normalized CE's after transaction costs, and the lower chart displays the outperformance ratios based upon them. The dotted lines in the upper chart represent EW. The results are averages across the datasets, D2-D13 (D1 is omitted due to its shorter sample period), and $T = 120$ is used.



(a) Certainty Equivalent



(b) Outperformance, P(>EW)

Figure 6: Sub-Period Performance with Calibrated δ: Utility Maximization

This figure visualizes the performance of the calibrated turnover aversion models in sub-periods (horizontal axis). Each sub-period is ten-year long and five-years apart from each other, except the last sub-period, SP13, which ends in 2015.12. The upper chart displays the normalized CE's after transaction costs, and the lower chart displays the outperformance ratios based upon them. The dotted lines in the upper chart represent EW. The results are averages across the datasets, D2-D13 (D1 is omitted due to its shorter sample period), and $T = 120$ is used.

even in utility maximization.

The empirical results favor the equal-weight portfolio as the reference portfolio. A natural question that arises would then be whether any fixed-weight portfolio could yield the same performance. While I do not test any other fixed-weight portfolios, there are a few reasons to favor the equal-weight portfolio over other fixed-weight portfolios. First of all, it is economically meaningful as it assumes that all the assets have the same return-risk ratio, which is in line with the capital asset pricing model. Secondly, if we randomly choose a portfolio under (strict) short-sale constraints, the expected portfolio is the equal-weight portfolio. Furthermore, if the number of assets increases, it converges to the equal-weight portfolio (See Appendix G for proof).

6 Concluding Remarks

Investors are reluctant to adopt an optimal portfolio when the parameters are subject to estimation errors. I introduce the turnover aversion utility which reflects this behavior and develop a portfolio model that maximizes it.

Turnover aversion is irrational behavior and maximizing turnover aversion utility results in a biased portfolio with an excessive weight on the reference portfolio. However, this portfolio is less volatile and performs robustly in the presence of parameter uncertainty. By incorporating turnover aversion, existing shrinkage estimators can also be improved significantly. This suggests that the usual method of maximizing the expected utility (without turnover aversion) is sub-optimal due to model parameter uncertainty.

Penalizing the turnover from the equal-weight portfolio renders substantially better performance than penalizing the turnover from the current portfolio even in the presence of transaction costs. Indeed, the former incurs lower transaction costs. This result is a sharp contrast to the widely-accepted belief that accounting for transaction costs improves performance and reduces transaction costs: this is perhaps true, but the effect appears rather trivial compared to using the equal-weight portfolio.

A data-driven calibration method to determine the optimal degree of turnover aversion is offered. This method is readily implementable and revealed to be very effective, generating superior performance when applied to various models and circumstances. Comprehensive robustness tests confirm that the above findings are valid over different sample periods, datasets, and optimization criteria.

Classical portfolio optimization models suffer from parameter uncertainty, and there are many advanced models that address this issue. But then again, these models face the challenge of quantifying uncertainty and estimating model parameters. The turnover aversion models perform superior by mitigating sensitivity to the "uncertainty of parameter uncertainty." Irrational, excessive aversion to turnover is indeed rational behavior in the uncertain world.

A Utility Maximization

A.1 Proof of Proposition 1

The expected utility can be rearranged as follows:

$$\begin{aligned} E[U(a,b)] = & \, aE[\hat{w}_{ml}]'\mu + bw_0'\mu - \frac{\gamma}{2}\left(a^2 E[\hat{w}_{ml}'\Sigma\hat{w}_{ml}] + 2abE[\hat{w}_{ml}]'\Sigma w_0 + b^2 w_0'\Sigma w_0\right) \\ & - \frac{\delta}{2}\left(a^2 E[\hat{w}_{ml}'\Sigma\hat{w}_{ml}] + 2a(b-1)E[\hat{w}_{ml}]'\Sigma w_0 + (b-1)^2 w_0'\Sigma w_0\right). \end{aligned} \tag{A.1}$$

Differentiating the expected utility with respect to a and b, the first order conditions are given by

$$\begin{aligned} \frac{\partial E[U(a,b)]}{\partial a} = & \, E[\hat{w}_{ml}]'\mu - a\gamma E[\hat{w}_{ml}'\Sigma\hat{w}_{ml}] - b\gamma E[\hat{w}_{ml}]'\Sigma w_0 \\ & - a\delta E[\hat{w}_{ml}'\Sigma\hat{w}_{ml}] - (b-1)\delta E[\hat{w}_{ml}]'\Sigma w_0 = 0, \end{aligned} \tag{A.2}$$

$$\begin{aligned} \frac{\partial E[U(a,b)]}{\partial b} = & \, w_0'\mu - a\gamma E[\hat{w}_{ml}]'\Sigma w_0 - b\gamma w_0'\Sigma w_0 \\ & - a\delta E[\hat{w}_{ml}]'\Sigma w_0 - (b-1)\delta w_0'\Sigma w_0 = 0. \end{aligned} \tag{A.3}$$

Solving for a and b,

$$a^* = \frac{1}{(\gamma+\delta)} \frac{E[\hat{w}_{ml}]'\mu - w_0'\mu \dfrac{E[\hat{w}_{ml}]'\Sigma w_0}{w_0'\Sigma w_0}}{E[\hat{w}_{ml}'\Sigma\hat{w}_{ml}] - E[\hat{w}_{ml}]'\Sigma w_0 \dfrac{E[\hat{w}_{ml}]'\Sigma w_0}{w_0'\Sigma w_0}}, \tag{A.4}$$

$$b^* = \frac{1}{(\gamma+\delta)} \frac{E[\hat{w}_{ml}]'\mu - w_0'\mu \dfrac{E[\hat{w}_{ml}'\Sigma\hat{w}_{ml}]}{E[\hat{w}_{ml}]'\Sigma w_0}}{E[\hat{w}_{ml}]'\Sigma w_0 - w_0'\Sigma w_0 \dfrac{E[\hat{w}_{ml}'\Sigma\hat{w}_{ml}]}{E[\hat{w}_{ml}]'\Sigma w_0}} + \frac{\delta}{\gamma+\delta}. \tag{A.5}$$

Since $\hat{\mu}$ and $\hat{\Sigma}$ are independent of each other and

$$\hat{\mu} \sim N\left(\mu, \frac{\Sigma}{K}\right), \quad \hat{\Sigma} \sim \mathcal{W}_N(T-1,\Sigma)\frac{1}{T}, \tag{A.6}$$

it follows that

$$E[\tilde{\Sigma}^{-1}] = \Sigma^{-1}, \quad E[\hat{\mu}\Sigma^{-1}\hat{\mu}] = \frac{N}{K} + \theta^2, \tag{A.7}$$

where $\theta^2 = \mu'\Sigma^{-1}\mu$. The latter equation is from $K\hat{\mu}\Sigma^{-1}\hat{\mu} \sim \chi^2_N(K\mu'\Sigma^{-1}\mu)$. Also, it can be shown that (see Kan and Zhou (2007) and the references therein)

$$E[\hat{\mu}'\tilde{\Sigma}^{-1}\Sigma\tilde{\Sigma}^{-1}\hat{\mu}] = c_1\left(\frac{N}{K} + \theta^2\right), \tag{A.8}$$

where $c_1 = \frac{(T-2)(T-N-2)}{(T-N-1)(T-N-4)}$. Then,

$$E[\hat{w}_{ml}] = \frac{1}{\gamma}\Sigma^{-1}\mu, \tag{A.9}$$

$$E[\hat{w}'_{ml}\Sigma\hat{w}_{ml}] = \frac{c_1}{\gamma^2}\left(\frac{N}{K} + \theta^2\right). \tag{A.10}$$

Substituting (A.9) and (A.10) into (A.4) and (A.5),

$$a^* = \frac{\gamma}{\gamma+\delta}a_0^*, \tag{A.11}$$

$$b^* = \frac{\gamma}{\gamma+\delta}b_0^* + \frac{\delta}{\gamma+\delta}, \tag{A.12}$$

where

$$a_0^* = \frac{\theta^2 - \psi^2}{c_1\left(\frac{N}{K} + \theta^2\right) - \psi^2}, \tag{A.13}$$

$$b_0^* = \frac{c_1\left(\frac{N}{K} + \theta^2\right) - \theta^2}{c_1\left(\frac{N}{K} + \theta^2\right) - \psi^2}\frac{1}{\gamma}\frac{w_0'\mu}{w_0'\Sigma w_0}, \tag{A.14}$$

$$\psi^2 = \mu_0'\Sigma^{-1}\mu, \quad \mu_0 = \frac{w_0'\mu}{w_0'\Sigma w_0}\Sigma w_0. \tag{A.15}$$

A.2 Estimation of a^* and b^*

Estimation of a^* and b^* involves estimation of θ^2, ψ^2, and w_{im}. For θ^2 and ψ^2, the method proposed by Kan and Zhou (2007) is adopted with modification for the different distributional assumption of $\hat{\mu}$.

- Estimation of θ^2

Since

$$K\hat{\mu}\Sigma^{-1}\hat{\mu} \sim \chi^2_N(K\mu'\Sigma^{-1}\mu), \quad \frac{\hat{\mu}'\Sigma^{-1}\hat{\mu}}{\hat{\mu}'(T\hat{\Sigma})^{-1}\hat{\mu}} \sim \chi^2_{T-N}, \tag{A.16}$$

and they are independent of each other, it follows that

$$\frac{K}{T}\hat{\mu}'\hat{\Sigma}^{-1}\hat{\mu} \sim \frac{N}{T-N}\mathcal{F}_{N,T-N}(K\mu'\Sigma^{-1}\mu), \tag{A.17}$$

where $\mathcal{F}$ is a noncentral F-distribution. Following the proof in the appendix of Kan and Zhou (2007), the estimate of θ^2 is given by

$$\tilde{\theta}^2 = \frac{(T-N-2)\hat{\theta}^2 - N}{K} + \frac{2(\hat{\theta}^2)^{N/2}(1+\hat{\theta}^2)^{-(T-2)/2}}{KB_{\hat{\theta}^2/(1+\hat{\theta}^2)}(N/2,(T-N)/2)}, \tag{A.18}$$

where

$$\hat{\theta}^2 = \frac{K}{T}\hat{\mu}'\hat{\Sigma}^{-1}\hat{\mu}, \tag{A.19}$$

and $B_x(a,b) = \int_0^x y^{a-1}(1-y)^{b-1}dy$ is an incomplete beta function.

- Estimation of ψ^2

  Since

$$\frac{K(w_0'\hat{\mu})^2}{w_0'\Sigma w_0} \sim \chi^2_1\left(\frac{K(w_0'\mu)^2}{w_0'\Sigma w_0}\right), \quad \frac{Tw_0'\hat{\Sigma}w_0}{w_0'\Sigma w_0} \sim \chi^2_{T-1}, \tag{A.20}$$

  and they are independent of each other, it follows that

$$\frac{K}{T}\frac{K(w_0'\hat{\mu})^2}{w_0'\hat{\Sigma}w_0} \sim \frac{1}{T-1}\mathcal{F}_{1,T-1}\left(\frac{K(w_0'\mu)^2}{w_0'\Sigma w_0}\right). \tag{A.21}$$

  The estimate of ψ^2 is then given by

$$\tilde{\psi}^2 = \frac{(T-3)\hat{\psi}^2 - 1}{K} + \frac{2(\hat{\psi}^2)^{1/2}(1+\hat{\psi}^2)^{-(T-2)/2}}{KB_{\hat{\psi}^2/(1+\hat{\psi}^2)}(1/2,(T-1)/2)}, \tag{A.22}$$

  where

$$\hat{\psi}^2 = \frac{K}{T}\frac{(w_0'\hat{\mu})^2}{w_0'\hat{\Sigma}w_0}. \tag{A.23}$$

- Estimation of w_{im}

  From $Tw_0'\hat{\Sigma}w_0 \sim w_0'\Sigma w_0 \cdot \chi^2_{T-1}$,

$$\frac{w_0'\Sigma w_0}{Tw_0'\hat{\Sigma}w_0} \sim \text{inv-}\chi^2_{T-1}, \tag{A.24}$$

and

$$E\left[\frac{1}{w_0'\hat{\Sigma}w_0}\right] = \frac{T}{T-3}\frac{1}{w_0'\Sigma w_0}. \tag{A.25}$$

Therefore, an unbiased estimate of w_{im} is given by

$$\tilde{w}_{im} = \frac{1}{\gamma}\frac{T-3}{T}\frac{w_0'\hat{\mu}}{w_0'\hat{\Sigma}w_0}w_0. \tag{A.26}$$

Simulation studies suggest that the optimal portfolio with the above estimates sometimes underperforms the more restricted model of Tu and Zhou (2011), especially when T is small. Meanwhile, assuming $w_{im} = \frac{c}{\gamma}w_0$ for some constant c appears to yield more robust performance. This can be justified as $\frac{w_0'\mu}{w_0'\Sigma w_0}$ should be constant when the returns are $i.i.d.$ Accordingly, $w_{im} = \frac{c}{\gamma}w_0$ with $c = 3$ rather than $\tilde{w}_{im}$ is used in the empirical studies of the paper.

A.3 Estimation of K

Let T_c denote a month during the sample period. For the first 119 months into the sample period, $K = T$ is assumed. When $T_c \geq 120$, the covariance matrix of $\hat{\mu}$, $\Sigma_{\hat{\mu}}$, is estimated employing the method of Lo and MacKinlay (1988):

$$\tilde{\Sigma}_{\hat{\mu}} = \frac{T_c + T - 1}{T_c - 1}\hat{\Sigma}_{\hat{\mu}}, \tag{A.27}$$

where $\hat{\Sigma}_{\hat{\mu}}$ is the ML estimate of $\Sigma_{\hat{\mu}}$. Let $\bar{\Sigma}$ denote the average of $\hat{\Sigma}$:

$$\bar{\Sigma} = \frac{1}{T_c}\sum_{t=1}^{T_c}\hat{\Sigma}_t, \tag{A.28}$$

where $\hat{\Sigma}_t$ is $\hat{\Sigma}$ at month t. K is determined so that the distance between $\frac{1}{K}\bar{\Sigma}$ and $\tilde{\Sigma}_{\hat{\mu}}$ is minimized. Defining the distance as the Frobenius norm of the lower triangular part of $(\bar{\Sigma} - K\tilde{\Sigma}_{\hat{\mu}})$, K is obtained from

$$K = \frac{v_1'v_2}{v_1'v_1}, \quad v_1 = vech(\tilde{\Sigma}_{\hat{\mu}}),\ v_2 = vech(\bar{\Sigma}), \tag{A.29}$$

where $vech(\cdot)$ is the half-vectorization operator.

B Variance Minimization

B.1 Proof of Proposition 2

Differentiating the expected variance with respect to a, the first order condition is given by

$$\frac{\partial E[V(a)]}{\partial a} = (1+\delta)aE[(\hat{w}_{mv} - w_0)'\Sigma(\hat{w}_{mv} - w_0)] + E[(\hat{w}_{mv} - w_0)'\Sigma w_0] = 0. \tag{B.1}$$

Solving for a,

$$a^* = \frac{1}{1+\delta}\frac{w_0'\Sigma w_0 - E[\hat{w}_{mv}]'\Sigma w_0}{E[\hat{w}_{mv}'\Sigma\hat{w}_{mv}] + w_0'\Sigma w_0 - 2E[\hat{w}_{mv}]'\Sigma w_0}. \tag{B.2}$$

From Kan and Smith (2008),

$$E[\hat{w}_{mv}] = \frac{\Sigma^{-1}1_N}{1_N'\Sigma^{-1}1_N}, \tag{B.3}$$

$$E[\hat{w}_{mv}'\Sigma\hat{w}_{mv}] = \frac{T-2}{T-N-1}\frac{1}{1_N'\Sigma^{-1}1_N}. \tag{B.4}$$

Substituting (B.3) and (B.4) into (B.2),

$$a^* = \frac{1}{1+\delta}\frac{\sigma_0^2 - \sigma_{mv}^2}{\sigma_0^2 - \left(1 - \frac{N-1}{T-N-1}\right)\sigma_{mv}^2}, \tag{B.5}$$

where $\sigma_0^2 = w_0'\Sigma w_0$ and $\sigma_{mv}^2 = w_{mv}'\Sigma w_{mv} = (1_N'\Sigma^{-1}1_N)^{-1}$ are the variances of w_0 and w_{mv}, respectively.

B.2 Estimation of a^*

Unbiased estimates of σ_0^2 and σ_{mv}^2 can be obtained as follows:

$$\tilde{\sigma}_0^2 = \frac{T}{T-1}w_0'\hat{\Sigma}w_0, \quad \tilde{\sigma}_{mv}^2 = \frac{T}{T-N}\frac{1}{1_N'\hat{\Sigma}^{-1}1_N}. \tag{B.6}$$

It can be seen that with the above estimates, $0 < a^* < 1$.

C Kan and Zhou (2007) Three-Fund Rule with Turnover Aversion

Consider a portfolio strategy of the form

$$w(a,b,c) = a\hat{w}_{ml} + b\tilde{w}_{mv} + cw_0, \tag{C.1}$$

where

$$\hat{w}_{ml} = \frac{1}{\gamma}\tilde{\Sigma}^{-1}\hat{\mu}, \quad \tilde{w}_{mv} = \frac{1}{\gamma}\tilde{\Sigma}^{-1}1_N. \tag{C.2}$$

The problem is to determine a, b, and c so that the expected utility is maximized:

$$\begin{aligned}\max_{a,b,c} E[U(a,b,c)] =& E\left[w(a,b,c)\mu - \frac{\gamma}{2}w(a,b,c)'\Sigma w(a,b,c)\right. \\ & \left. -\frac{\delta}{2}(w(a,b,c) - w_0)'\Sigma(w(a,b,c) - w_0)\right].\end{aligned} \tag{C.3}$$

Differentiating the expected utility with respect to the model parameters, the first order conditions are given by

$$\frac{\partial E[U(a,b,c)]}{\partial a} = B_1 - a(\gamma+\delta)A_1 - b(\gamma+\delta)A_{12} - c(\gamma+\delta)A_{01} + \delta A_{01} = 0, \tag{C.4}$$

$$\frac{\partial E[U(a,b,c)]}{\partial b} = B_2 - a(\gamma+\delta)A_{12} - b(\gamma+\delta)A_2 - c(\gamma+\delta)A_{02} + \delta A_{02} = 0, \tag{C.5}$$

$$\frac{\partial E[U(a,b,c)]}{\partial c} = B_0 - a(\gamma+\delta)A_{01} - b(\gamma+\delta)A_{02} - c(\gamma+\delta)A_0 + \delta A_0 = 0, \tag{C.6}$$

where

$$B_0 = E[w_0'\mu] = w_0'\mu, \tag{C.7}$$
$$B_1 = E[w_{ml}'\mu] = \frac{1}{\gamma}\theta^2, \tag{C.8}$$
$$B_2 = E[w_{mv}'\mu] = \frac{1}{\gamma}1_N'\Sigma^{-1}\mu, \tag{C.9}$$
$$A_0 = E[w_0'\Sigma w_0] = w_0'\Sigma w_0, \tag{C.10}$$
$$A_1 = E[w_{ml}'\Sigma w_{ml}] = \frac{c_1}{\gamma^2}\left(\frac{N}{K} + \theta^2\right), \tag{C.11}$$
$$A_2 = E[w_{mv}'\Sigma w_{mv}] = \frac{c_1}{\gamma^2}1_N'\Sigma^{-1}1_N, \tag{C.12}$$
$$A_{01} = E[w_0'\Sigma w_{ml}] = \frac{1}{\gamma}w_0'\mu, \tag{C.13}$$
$$A_{02} = E[w_0'\Sigma w_{mv}] = \frac{1}{\gamma}w_0'1_N, \tag{C.14}$$
$$A_{12} = E[w_{ml}'\Sigma w_{mv}] = \frac{c_1}{\gamma^2}1_N'\Sigma^{-1}\mu. \tag{C.15}$$

Solving for a, b, and c,

$$a^* = \frac{1}{\gamma+\delta}a_0^*, \tag{C.16}$$
$$b^* = \frac{1}{\gamma+\delta}b_0^*, \tag{C.17}$$
$$c^* = \frac{1}{\gamma+\delta}\frac{B_1 - a_0^*A_1 - b_0^*A_{12}}{A_{01}} + \frac{\delta}{\gamma+\delta}, \tag{C.18}$$

where

$$a_0^* = \frac{B_1(A_0A_2 - A_{02}^2) - B_2(A_0A_{12} - A_{01}A_{02}) - B_0(A_{01}A_2 - A_{02}A_{12})}{A_0A_1A_2 - A_1A_{02}^2 - A_0A_{12}^2 - A_2A_{01}^2 + 2A_{01}A_{02}A_{12}}, \tag{C.19}$$
$$b_0^* = \frac{B_1(A_01A_02 - A_0A_{12}) - B_2(A_{01}^2 - A_0A_1) - B_0(A_1A_{02} - A_{01}A_{12})}{A_0A_1A_2 - A_1A_{02}^2 - A_0A_{12}^2 - A_2A_{01}^2 + 2A_{01}A_{02}A_{12}}. \tag{C.20}$$

D Moments of Optimal Portfolio Weights

D.1 Proof of Proposition 3

When $w_0 = w_{t-}$, the turnover aversion portfolio at time t, w_t^c, can be written as

$$\begin{aligned} w_t^c &= (1-\alpha)w_t^* + \alpha w_{t-1}^c \\ &= (1-\alpha)w_t^* + \alpha((1-\alpha)w_{t-1}^* + \alpha w_{t-2}^c) \\ &\vdots \\ &= (1-\alpha)(w_t^* + \alpha w_{t-1}^* + \cdots + \alpha^{t-1} w_1^*) + \alpha^t w. \end{aligned} \tag{D.1}$$

Since the returns are *i.i.d.*, $E(w_t^*) = E(w_{t-1}^*) = \cdots = E(w_1^*)$, and it follows that

$$E(w_t^c) = (1-\alpha^t)E(w_t^*) + \alpha^t w, \tag{D.2}$$

$$V(w_{it}^c) = (1-\alpha)^2 V(w_{it}^* + \alpha w_{it-1}^* + \cdots + \alpha^{t-1} w_{i1}^*), \tag{D.3}$$

where w_{it} denotes the i-th element of w_t.

When $w_0 = w_{ew}$, the turnover aversion portfolio at time t, w_t^e, has the form

$$w_t^e = (1-\alpha)w_t^* + \alpha w_{ew}, \tag{D.4}$$

and its moments are given by

$$E(w_t^e) = (1-\alpha)E(w_t^*) + \alpha w_{ew}, \tag{D.5}$$

$$V(w_{it}^e) = (1-\alpha)^2 V(w_{it}^*). \tag{D.6}$$

As $0 < Cov(w_{it}^*, w_{it-1}^*) < 1$ for $t > 1$,

$$(1-\alpha)^2(1+\alpha^2+\cdots+\alpha^{2(t-1)})V(w_{it}^*) < V(w_{it}^c) < (1-\alpha^t)^2 V(w_{it}^*). \tag{D.7}$$

Therefore,

$$V(w_{it}^e) < V(w_{it}^c) < V(w_{it}^*). \tag{D.8}$$

D.2 Proof of Proposition 4

Note that

$$\Delta w_{it}^c = (1-\alpha)\Delta w_{it}^* + \alpha \Delta w_{it-1}^c, \tag{D.9}$$

$$\Delta w_{it}^e = (1-\alpha)\Delta w_{it}^*. \tag{D.10}$$

Therefore,

$$E\left[(\Delta w_{it}^c)^2\right] = (1-\alpha)^2 E\left[(\Delta w_{it}^*)^2\right] + \alpha^2 E\left[(\Delta w_{it-1}^c)^2\right] + 2\alpha(1-\alpha)E\left[\Delta w_{it}^* \Delta w_{it-1}^c\right], \tag{D.11}$$

$$E\left[(\Delta w_{it}^e)^2\right] = (1-\alpha)^2 E\left[(\Delta w_{it}^*)^2\right]. \tag{D.12}$$

If $E\left[\Delta w_{it}^* \Delta w_{it-1}^c\right] > -\frac{\alpha}{2(1-\alpha)} E\left[(\Delta w_{it-1}^c)^2\right]$,

$$E\left[(\Delta w_{it}^e)^2\right] < E\left[(\Delta w_{it}^c)^2\right]. \tag{D.13}$$

Since $\Delta w_{i1}^c = (1-\alpha)(w_{i1}^* - w_i)$ and $\Delta w_{i1}^* = (w_{i1}^* - w_i)$, $E\left[(\Delta w_{i1}^c)^2\right] < E\left[(\Delta w_{i1}^*)^2\right]$. From (D.9), it follows that

$$E\left[(\Delta w_{it}^c)^2\right] < E\left[(\Delta w_{it}^*)^2\right]. \tag{D.14}$$

E Treatment of Negative Weights on the Risky Portfolio

While financial theories do not allow negative expected returns associated with positive risks, it is not uncommon to have negative mean estimates. This can lead to a negative optimal weight on the risky portfolio as illustrated in Figure E.1. In this event, the global minimum-variance portfolio will have a negative expected return, and the utility maximizing portfolio will short the tangent portfolio P_T and invest the proceeds in the risk-free asset, thus being placed somewhere on the upper part of the dashed line. A tangent portfolio with a positive slope does not exist in this case, and the usual method to find the tangent portfolio, *i.e.*, dividing the risky asset weights by their sum will lead to P_T, which has the minimum Sharpe ratio and thereby is worse off than any other portfolios in the feasible set.

Suppose the vertical dotted line indicates the target variance, $\sigma_{\max}^2$, and the variance of the tangent portfolio is σ_T^2. If the tangent portfolio weights are multiplied by $\sigma_{\max}/\sigma_T$, the resulting portfolio will be P_1, whereas the optimal portfolio should be P_1' which can be obtained with the multiplication factor, $-\sigma_{\max}/\sigma_T$. Similarly, the variance targeting portfolio based on the global minimum-variance portfolio should be P_2' rather than P_2. This paper adopts this approach.

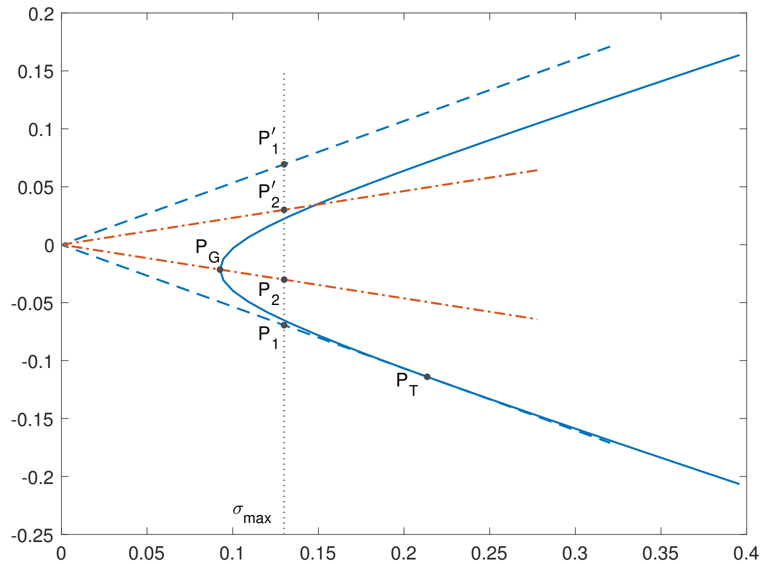


Figure E.1: Variance Targeting When the Tangent Portfolio Has a Negative Expected Return

This graph illustrates the minimum-variance portfolios when the tangent portfolio has a negative expected return. P_T and P_G respectively denote the tangent portfolio and the global minimum-variance portfolio. The dashed line represents the feasible set that can be obtained from P_T, whereas the dash-dot line represents the feasible set from P_G. P_1 and P_1' (P_2 and P_2') are the portfolios derived from P_T (P_G) that satisfy the variance target $\sigma^2_{\max}$.

F Detailed Analysis of Empirical Results

This section provides detailed analyses of the empirical results. Analyses are primarily based on the results presented in Table 8 and 9 but also refer to other results such as those for $T = 60$ and 120, which can be found in Internet Appendix.

ML vs. TZML vs. TAML Under variance targeting, ML outperforms EW in ten datasets and has a considerably higher mean CE before transaction costs. While much of the advantage disappears after transaction costs, it still outperforms EW in nine datasets. The benefit of combining ML with EW is evident from TZML and TAML. The mean CE's of TZML and TAML(0) after transaction costs are respectively 0.613 and 0.623, whereas those of ML and EW are respectively 0.489 and 0.243. TZML and TAML(0) outperform EW in all but one datasets before transaction costs and in ten datasets after transaction costs. The improvement over ML is particularly noticeable when T is small (see Internet Appendix). Unlike the superior performance of TAML(0) to TZML observed in the simulation studies, they perform comparably when tested on the real market data.

KZ vs. TZML vs. TZKZ Consistent with the findings of Tu and Zhou (2011), TZKZ outperforms TZML and KZ. This can be anticipated to some extent as TZKZ involves three portfolios, whereas the others involve only two. Between KZ and TZML, TZML yields higher CE's, whilst KZ outperforms EW more often. Overall, KZ and TZML perform comparably well when T is large, but the performance of KZ deteriorates rapidly as T decreases. This can be attributed to the fact that both ML and MV which comprise KZ depend on the input parameters and therefore subject to estimation errors. As discussed below, the performance of KZ improves markedly when it is augmented with turnover aversion.

Effects of Turnover Aversion Penalizing turnover from w_{ew} improves portfolio performance in most cases regardless of the base model. Improvements are particularly noticeable in the presence of transaction costs and in terms of the outperformance ratio: among the models with similar CE's, those incorporating turnover aversion tend to outperform EW more frequently. Among KZ, TZML, TZKZ, and TAML, KZ benefits most by incorporating turnover aversion. This is because the original KZ, contrary to the others, does not involve w_{ew}. Comparing KZ with TZKZ , the best performing KZ usually outperforms the best performing TZKZ even though both models involve the same three portfolios: *e.g.*, in Table 8, the best performing KZ (KZ(3)) and TZKZ (TZKZ(2)) have the mean CE after transaction costs of 0.701 and 0.690, respectively.

The models with $w_0 = w_{t-}$, *i.e.*, KZc, TZMLc, TZKZc, and TAMLc, perform rather disappointingly. When these are compared with their counterparts with $w_0 = w_{ew}$, the latter models almost always perform better with respect to all criteria. TAMLc performs particularly poor: TAMLc(1) underperforms TAML(0) even after transaction costs. This suggests that the estimation errors in the actual market data are substantially higher than assumed: in simulations, TAMLc outperformed TAML when estimation errors were small ($T = 120$ or 240) but was outperformed otherwise (T=60).

The above finding conveys an important message as the turnover aversion models with $w_0 = w_{t-}$ are similar to the models that take transaction costs into account during optimization (*e.g.*, Gârleanu and Pedersen, 2013; DeMiguel et al., 2015; Olivares-Nadal and DeMiguel, 2015). Shrinking towards the current portfolio does improve portfolio performance but is less effective than shrinking towards the equal-weight portfolio regardless of the presence of transaction costs. It appears that a certain degree of robustness should be assured beforehand in order to benefit from the former approach.

Estimation of K In TAMLK, K is estimated using the method described in Appendix A.3. TAMLK(0) and TAMLK(1) respectively outperform TAML(0) and TAML(1) regardless of T, but the improvement is more prominent when $\delta = 0$. Besides, TAMLK performs more consistently across the datasets and outperform EW more frequently. This suggests that the proposed method addresses the uncertainty in mean more adequately than the simple assumption of $K = T$. With the better assessment of estimation errors, the δ required to maximize performance would become smaller. This explains why the performance improvement is more pronounced when $\delta = 0$.

TML and TMV TML and TMV respectively outperform ML and MV as well as EW. They seem to share the characteristics of TAML and TAMV. However, with the tolerance factor, $\tau = 0.05$, they do not perform particularly well compared to the turnover aversion models. Although not pursued here, research on determining optimal τ must be worthwhile.

MV vs. TAMV vs. VT Both MV and TAMV(0) perform poorly and are among the worst performers. This is an unexpected result since MV is known to demonstrate robust performance: see *e.g.*, DeMiguel et al. (2009); Han (2016). The poor performance mainly stems from the datasets; D6, D7, D8, and D9, where the global minimum-variance portfolio often has a negative expected return. When the expected return is negative, as explained in Appendix E, the optimal portfolio becomes P_2' which has a negative exposure to the global minimum-variance portfolio. While, in principle, P_2' should outperform P_2, P_2 is found to perform better and its performance is comparable to that of EW (unreported). This explains why these models perform poorly as opposed to what has been reported in the literature. The results from utility maximization where variance minimizing portfolios are unadjusted also confirm this: they perform comparably to or outperform EW. Although negative expected returns are allowed in this paper to reveal potential issues, it would be best in practice to prevent such cases in the first place by either using an alternative estimator or excluding negative return assets.

Contrary to the simulation results, incorporating turnover aversion (TAMV(1)) improves performance significantly. This indicates that the actual estimation error of the covariance matrix is larger than assumed.

The short-sale constrained models perform better than their counterparts especially in terms of the outperformance ratio: TAMV+ in particular has a much higher CE and outperforms EW in all but one datasets when $T = 120$ in variance targeting. VT also performs robustly. This is because VT is implicitly short-sale constrained and depends only on the cross-sectional variation of the variances, which is stable over time. Nonetheless, overall performance of the variance minimization models is not impressive.

Effects of Short-sale Constraint When optimal portfolio models are subject to the short-sale constraint, they perform robustly and outperform EW more frequently compared with their unconstrained counterparts. In fact, the short-sale constrained models are ranked at the top in terms of the outperformance ratio and demonstrate robust performance in utility maximization. They also have significantly lower turnover and leverage. Nevertheless, their performance is rather suppressed as evidenced by the low CE's. The short-sale constraint is an effective tool to enhance robustness especially when parameter uncertainty is large, but at the same time it hinders high return potential.

Effects of Estimation Window Size Asset returns are not stationary over a long period, and using a large estimation window does not necessarily lead to better moment estimates. Determining the optimal estimation window size depends mainly on two aspects: estimation errors and transaction costs.

Based on the performance before transaction costs, many portfolio models turn out to perform best when $T = 120$ and worst when $T = 60$. Exceptions are the variance-minimization models which perform best when $T = 240$. It appears that the mean estimation error declines with T until some point and then increases again, whereas the covariance estimation error continues to decline with T at least within the range considered here. The window size also has an effect on the portfolio loadings of the shrinkage estimators: a larger T will put more weight on $\hat{w}_{ml}$ regardless of the actual estimation errors, and their performance could deteriorate rapidly beyond a certain T. On the other hand, a larger window size is always beneficial in terms of transaction costs as the moment estimates become more persistent leading to lower turnover.

The window size will have to be determined considering several factors such as the portfolio strategy, actual transaction costs, and dataset. Nevertheless, for the datasets considered in this paper, $T = 120$ seems to be a reasonable choice especially for the turnover aversion models. Although not pursued in this paper, applying different window sizes to mean and covariance estimations may improve overall estimation accuracy.

Performance of EW While EW is known to perform superior, the empirical results in this paper are rather contrary. EW is one of the worst performing portfolios in terms of mean CE. In addition, most other portfolios outperform EW in more than half of the datasets when $T = 120$ or 240 in variance targeting. This unexpected result is to some extent related to the variance constraint. Optimal portfolios generally perform well and robustly under the variance constraint, but many of them are outperformed by (unadjusted) EW without the constraint. If EW is adjusted so as to maximize utility, its performance also drops considerably and it is outperformed by many optimal portfolios again. Still, optimal portfolios perform relatively worse without the variance constraint.

G Random Portfolio Convergence

Consider a portfolio comprised of N assets. The portfolio weights, $\{w_i | w_1 + \ldots + w_N = 1,\ w_i > 0\}$, form an $(N-1)$-dimensional simplex and follow a Dirichlet distribution of order N:

$$f(w_1, \ldots, w_N) = \frac{1}{\mathcal{B}(\alpha_1, \ldots, \alpha_N)} \prod_{i=1}^{N} w_i^{\alpha_i - 1}. \tag{G.1}$$

If the weights are randomly chosen from a uniform distribution, $\alpha_1 = \alpha_2 = \ldots = \alpha_N = 1$, and the first moment of the i-th weight is given by

$$E(W_i) = \frac{\Gamma(1+1)\Gamma(N)}{\Gamma(N+1)\Gamma(1)} = \frac{1}{N}, \tag{G.2}$$

and the second moments are given by

$$E\left(W_i^2\right) = \frac{\Gamma(1+2)\Gamma(N)}{\Gamma(N+2)\Gamma(1)} = \frac{2}{(N+1)N}, \tag{G.3}$$

$$E(W_i W_j) = \frac{\Gamma(1+1)\Gamma(1+1)\Gamma(N)}{\Gamma(N+2)\Gamma(1)\Gamma(1)} = \frac{1}{(N+1)N}. \tag{G.4}$$

Therefore, the random portfolio converges to the equal-weight portfolio as N increases.

References

Avramov, D., Zhou, G., 2010. Bayesian portfolio analysis. The Annual Review of Financial Economics 2 (1), 25–47.

Bessler, W., Opfer, H., Wolff, D., 2014. Multi-asset portfolio optimization and out-of-sample performance: an evaluation of Black–Litterman, mean-variance, and naïve diversification approaches. The European Journal of Finance, 1–30.

Black, F., Litterman, R., 1992. Global portfolio optimization. Financial Analysts Journal 48 (5), 28–43.

Brown, S. J., 1976. Optimal portfolio choice under uncertainty: A bayesian approach. Ph.D. thesis, University of Chicago, Graduate School of Business.

Brown, S. J., 1978. The portfolio choice problem: comparison of certainty equivalence and optimal bayes portfolios. Communications in Statistics-Simulation and Computation 7 (4), 321–334.

Ceria, S., Stubbs, R. A., 2016. Incorporating estimation errors into portfolio selection: Robust portfolio construction. In: Asset Management. Springer, pp. 270–294.

DeMiguel, V., Garlappi, L., Uppal, R., 2009. Optimal versus naive diversification: How inefficient is the 1/N portfolio strategy? Review of Financial Studies 22 (5), 1915–1953.

DeMiguel, V., Martín-Utrera, A., Nogales, F. J., 2015. Parameter uncertainty in multiperiod portfolio optimization with transaction costs. Journal of Financial and Quantitative Analysis 50 (06), 1443–1471.

Fabozzi, F. J., Kolm, P. N., Pachamanova, D. A., Focardi, S. M., 2007. Robust portfolio optimization and management. John Wiley & Sons.

Gârleanu, N., Pedersen, L. H., 2013. Dynamic trading with predictable returns and transaction costs. The Journal of Finance 68 (6), 2309–2340.

Goldfarb, D., Iyengar, G., 2003. Robust portfolio selection problems. Mathematics of Operations Research 28 (1), 1–38.

Han, C., 2016. Improving the naïve portfolio strategy. Working Paper.

Jagannathan, R., Ma, T., 2003. Risk reduction in large portfolios: Why imposing the wrong constraints helps. The Journal of Finance 58 (4), 1651–1684.

Jorion, P., 1986. Bayes-Stein estimation for portfolio analysis. The Journal of Financial and Quantitative Analysis 21 (3), 279–292.

Kan, R., Smith, D. R., 2008. The distribution of the sample minimum-variance frontier. Management Science 54 (7), 1364–1380.

Kan, R., Wang, X., Zhou, G., 2016. On the value of portfolio optimization in the presence of estimation risk: the case with and without risk-free asset. Working Paper.

Kan, R., Zhou, G., 2007. Optimal portfolio choice with parameter uncertainty. Journal of Financial and Quantitative Analysis 42 (03), 621–656.

Kirby, C., Ostdiek, B., 2012. It's all in the timing: Simple active portfolio strategies that outperform naïve diversification. The Journal of Financial and Quantitative Analysis 47 (2), 437–467.

Klein, R. W., Bawa, V. S., 1976. The effect of estimation risk on optimal portfolio choice. Journal of Financial Economics 3 (3), 215–231.

Ledoit, O., Wolf, M., 2004. Honey, I shrunk the sample covariance matrix. The Journal of Portfolio Management 30 (4), 110–119.

Lo, A. W., MacKinlay, A. C., 1988. Stock market prices do not follow random walks: Evidence from a simple specification test. Review of financial studies 1 (1), 41–66.

Markowitz, H., 1952. Portfolio selection. The Journal of Finance 7 (1), 77–91.

Olivares-Nadal, A. V., DeMiguel, V., 2015. A robust perspective on transaction costs in portfolio optimization. Working Paper.

Pástor, L., 2000. Portfolio selection and asset pricing models. The Journal of Finance 55 (1), 179–223.

Pástor, L., Stambaugh, R. F., 2000. Comparing asset pricing models: an investment perspective. Journal of Financial Economics 56 (3), 335–381.

Tu, J., Zhou, G., 2011. Markowitz meets Talmud: A combination of sophisticated and naive diversification strategies. Journal of Financial Economics 99 (1), 204–215.